Problem 1. If $z \notin \bar{B}(x,r)$, then $d(x,z) > r$, so $\delta = d(x,z) - r >$

If $y \in B(z,\delta)$, then

$$d(x,z) \le d(x,y) + d(y,z) < d(x,y) + \delta$$

hence $d(x,y) > d(x,z) - \delta = r$. This shows that $z \notin \overline{B(x,r)}$,
hence $\overline{B(x,r)} \subset \bar{B}(x,r)$.

In the discrete metric on any space X, we have for all $x \in X$.

$$\overline{B(x,1)} = \{x\}$$
$$\bar{B}(x,1) = X$$

Problem 2 · X infinite set with the cofinite topology
· Y a Hausdorff space

Then $\mathcal{C}(X,Y)$ consists of the constant functions.

Proof: If f is constant, i.e. $f \equiv c$, then

$$f^{-1}(A) = \begin{cases} \emptyset & \text{if } c \notin A \\ X & \text{if } c \in A \end{cases}$$

which is always open. Hence the inverse image of any set is open and f is continuous. (This is actually always true)
If f is not constant, then there are $a, b \in f(X)$ with $a \neq b$. Let V_1, V_2 be disjoint neighbourhoods of a and b. Then $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ are disjoint and open, i.e. both are cofinite. But an infinite set cannot have disjoint cofinite sets.

Problem 3 $f: X \to Y$

f is open $\Leftrightarrow$ $f(\text{Int } A) \subset \text{Int } f(A)$ for all subsets A of X

Proof: $\Rightarrow$ If f is open, $f(\text{Int } A)$ is open and contained in $f(A)$. Since $\text{Int } f(A)$ is the largest open set contained in $f(A)$, we must have $f(\text{Int } A) \subset \text{Int } f(A)$

$\Leftarrow$ If U is open, then $f(U) = f(\text{Int } U) \subset \text{Int } f(U)$. But a set contained in its own interior is open.

f is closed $\Leftrightarrow$ $\overline{f(A)} \subset f(\bar{A})$ for all subsets A of X

Proof: The proof is the same. $\Rightarrow$ If f is closed, then $f(\bar{A})$ is closed and contains $f(A)$. Since $\overline{f(A)}$ is the smallest set containing $f(A)$, we have $\overline{f(A)} \subset f(\bar{A})$

$\Leftarrow$ If E is closed, then $f(E) = f(\bar{E}) \supset \overline{f(E)}$. But a set containing its own closure is closed.

Problem 4 $p: X \to Y$ and $s: Y \to X$ are continuous such that $p \circ s = \text{id}$.

a) p is a quotient map

Proof: p is surjective since $p \circ s$ is surjective.

If $V \subset Y$ and $p^{-1}(V)$ is open, then $V = (p \circ s)^{-1}(V)$ $= s^{-1}(p^{-1}(V))$ is open since s is continuous. Hence p is a quotient map.

b) s is an imbedding

Proof: We shall prove that $s: Y \to s(Y)$ is a homeomorphism i.e. it is injective, continuous and s^{-1} is continuous.

s is injective, since if $s(y) = s(y')$, then $y = p(s(y)) = p(s(y')) = y'$. It is also given that s is continuous.

Finally, σ^{-1} is continuous, since $\sigma^{-1} = p|_{\sigma(Y)}$ and p is continuous.

Problem 5 $f: X \to Y$ continuous, X and Y locally compact Hausdorff spaces. f is extended to $f: X^* \to Y^*$ by $f(\infty) = \infty$. Then f^* is continuous $\Leftrightarrow$ $f^{-1}(K)$ is compact in X for all compact $K \subset Y$. (Such maps are called proper)

Proof:

$\Rightarrow$ If f^* is continuous and $K \subset Y$ compact, then $Y^* \setminus K$ is a neighbourhood of ∞ in Y, so $V = f^{-1}(Y^* \setminus K)$ is a neighbourhood of ∞ in X. Therefore $f^{-1}(K) = X^* \setminus V$ is compact.

$\Leftarrow$ If U is a nbh of ∞ in Y, then $K = Y^* \setminus U$ is compact. Hence $f^{-1}(K)$ is compact in X, so $f^{-1}(U) = X^* \setminus f^{-1}(K)$ is a nbh of ∞ in X.

Problem 6 A, B compact in X and Y, $W \subset X \times Y$ an open set containing $A \times B$. Then there are open sets U, V containing A and B such that $U \times V \subset W$

Proof:

Fix $a \in A$. Then since W is open, there is for every $b \in B$ neighbourhoods U_b of a and V_b of b such that $U_b \times V_b \subset W$. The sets $\{V_b\}_{b \in B}$ cover B, hence by compactness, there are $b_1, \dots, b_n \in B$ such that $V_{b_1}, \dots, V_{b_n}$ cover B. Let $U_a = \bigcap_{i=1}^{n} U_{b_i}$ and $V_a = \bigcup_{i=1}^{n} V_{b_i}$.

Then U_a is a nbh of a, V_a a nbh of B and $U_a \times V_a \subset W$.

The sets $\{U_a\}_{a\in A}$ cover A, hence by compactness, there are $a_1, \cdots, a_m \in A$ such that $U_{a_1}, \cdots, U_{a_m}$ cover A.

Let $U = \bigcup_{i=1}^{m} U_{a_i}$ and $V = \bigcap_{i=1}^{m} V_{a_i}$

Then U is a nbh of A, V a nbh of B and $U \times V \subset W$

<u>Problem 7</u> x is an accumulation point of (x_k) if every nbh V of x contains x_k for infinitely many k.

x is an accumation point for $(x_k) \iff x \in \bigcap_m F_m$ where

$$F_m = \overline{\{x_k \mid k \geq m\}}$$

Proof: $\Rightarrow$ Let x be an accumulation point and $m \in \mathbb{N}$. If V is a nbh of x, then $x_k \in V$ for infinitely many k, hence for some $k \geq m$. This proves that $V \cap \{x_k \mid k \geq m\} \neq \emptyset$ so $x \in F_m$. Since m was arbitrary, $x \in \bigcap_m F_m$.

$\Leftarrow$ Let V be a nbh of x. Since $x \in F_1$, we have $V \cap \{x_k \mid k \geq 1\} \neq \emptyset$ and we may pick m_1 such that $x_{m_1} \in V$. Since $x \in F_{m_1+1}$, we have $V \cap \{x_k \mid k \geq m_1+1\} \neq \emptyset$ and we may pick $m_2 > m_1$ such that $x_{m_2} \in V$. Continuing like this, we find a strictly increasing sequence of integers (m_k) such that $x_{m_k} \in V$. This proves that x is an accumulation point.

The set of accumulation points is closed since it is an intersection of closed sets.

If X is also compact it must be nonempty, since this is a decreasing family of compacts.

<u>Problem 8</u> If $f: X \to Y$ is a quotient map, Y is connected and $f^{-1}(y)$ is connected for all $y \in Y$, then X is connected.

Proof:

Suppose $A \subset X$ is a nonempty subset which is both open and closed. We will prove that $A = X$. If $y \in f(A)$, then $f^{-1}(y) \subset A$, otherwise $A \cap f^{-1}(y)$, $A^c \cap f^{-1}(y)$ would be a separation of $f^{-1}(y)$. Hence $A = f^{-1}(f(A))$ and $f(A)$ must be both open and closed since f is a quotient map. But then we must have $f(A) = Y$ since Y is connected and $A = f^{-1}(Y) = X$.