

Topology (MAT3500/4500) 2022.

Practical info

- Mandatory assignment : required to take final exam
problems available late Sept.
submit solution by PDF on Canvas
(use LaTeX)
deadline late Oct.
- Final exam : written, Dec. 6.
- Schedule
Mon. 14-16, Thn. 12-14 lecture (M. Y.)
Wed 10-12 exercise (Ola Sande)
- Announcements
Course website (see MAT4500)
Canvas.
- Main reference.
Munkres, "Topology : a first course"

Introduction to main concepts

This course is about

- abstract topological spaces
- fundamental groups.

Abs. top. sp. is a framework to understand

- approximation
- continuity,

beyond basic examples like lines, planes, ...

Benefits

topology

- geometry : foundation for higher dimensional shapes (manifold, spaces in alg. top. ...)
- analysis : approx. for operations on functions, ∞ -dim spaces
- algebra : geometric intuition for algebraic equations

For example

- boundary at ∞ : what are the limits of random walks?
- commutative rings as "polynomial functions" reverse the viewpoint of "polynom. funcs. form a comm. ring"

Fund. grp. : first algebraic invariant of top. sp.

Basic idea : count loops in a space, (with a fixed start/end point) but don't care about continuous deform.

For example : Space $\mathbb{R}^2 \setminus \{o = (0,0)\}$



$\leadsto$ only care about "how many times does it go around o?"

$\leadsto$ "following two loops one after another" defines a group law.

This goes far : analogy with Galois theory, ...

§ 20 Metric spaces

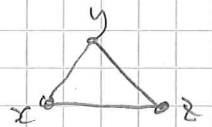
Basic idea: use distance between points to understand approx.

Def. X : set ; a metric (or distance func.) on X is a map $d: X \times X \rightarrow \mathbb{R}$ s.t.

$$1) d(x, y) \geq 0, \text{ eq. holds iff } x = y.$$

$$2) d(x, y) = d(y, x)$$

$$3) d(x, y) + d(y, z) \geq d(x, z)$$



Also (X, d) is a metric space

Ex. $X = \mathbb{R}^n$, $d(x, y) = \sqrt{\sum_{1 \leq i \leq n} (x_i - y_i)^2}$

another choices: $d(x, y) = \sum |x_i - y_i|$, $\max_i |x_i - y_i|$

Def. (X, d) met sp. $x \in X$, $\varepsilon > 0$

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\} \quad \begin{array}{l} \varepsilon\text{-ball centered} \\ \text{at } x \end{array}$$

(also $B(x, \varepsilon)$, $B_\varepsilon(x)$, ...)

Def. $U \subset X$ is open if $\forall x \in U \exists \varepsilon > 0$ s.t. $B_d(x, \varepsilon) \subset U$.

U open: if $x \in U$ and y is "close enough" to x , then $y \in U$ how close depends on x .

Ex. $B_d(x, \varepsilon)$ is open.

Th'm 20.3 $X = \mathbb{R}^n$, $d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$
 $\rho(x, y) = \max_i |x_i - y_i|$

$U \subset \mathbb{R}^n$ is open for $d \Leftrightarrow U$ is open for ρ

Proof key point $\rho(x, y) \leq d(x, y) \leq \sqrt{n} \rho(x, y)$

then $B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon) \subset B_d(x, \sqrt{n} \varepsilon)$

so U open for p , $x \in U$:

$\Rightarrow$ find ε s.t. $B_p(x, \varepsilon) \subset U$

$\Rightarrow$ above $B_d(x, \varepsilon) \subset U$ i.e. U open for d . $\square$

§ 18 Continuous functions.

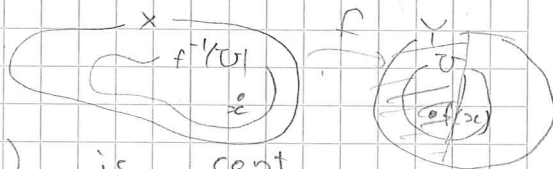
(this time only for metric spaces)

Def. (X, d_X) , (Y, d_Y) met-sp.

$f: X \rightarrow Y$ is continuous if $\forall x, \delta > 0 \exists \varepsilon > 0$

(#) s.t. $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$

f is cont. if x & x' close enough $\Rightarrow f(x)$ & $f(x')$ close enough.



Th'm $f: (X, d_X) \rightarrow (Y, d_Y)$ is cont

$\Leftrightarrow \forall U \subset Y$ open $f^{-1}(U) = \{x: f(x) \in U\}$ is open.

Proof $\Rightarrow$: fix open U , $x \in f^{-1}(U)$, want ε s.t.

$B_{d_X}(x, \varepsilon) \subset f^{-1}(U)$.

$f(x) \in U \Rightarrow \exists \delta: B_{d_Y}(f(x), \delta) \subset U$

U open

$\Rightarrow \exists \varepsilon$ s.t. (#) holds. ; use this ε .

$\Leftarrow$: fix x, δ , want ε s.t. (#) holds.

consider $U = B_{d_Y}(f(x), \delta)$. this is open.

so $f^{-1}(U)$ is open by assumption

$x \in f^{-1}(U) \Rightarrow \exists \varepsilon B_{d_X}(x, \varepsilon) \subset f^{-1}(U)$.

use this ε .

$\square$