

§ 18 for met. sp. cont'd.

what can be modelled by cont. maps?

- curves in plane / space $f: \mathbb{R} \rightarrow \mathbb{R}^2$ (or $\mathbb{R}^3$)
- cont. funcs. in many vars $\leftrightarrow f: \mathbb{R}^n \rightarrow \mathbb{R}$
 $f(x_1, \dots, x_n)$
- vector fields (distributions of vectors)
 $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ($v_{x,y} = f(x,y)$ at $(x,y) \in \mathbb{R}^2$)

Def. (X, d) met. sp., $A \subset X$ subset.

The closure of A is

$$\bar{A} = \{x \in X : \forall \varepsilon > 0 \exists a \in A \ d(x, a) < \varepsilon\}$$

"we can approx. x by points of A "

A is closed if $\bar{A} = A$.

Th'm (part of 18.1) $(X, d_x), (Y, d_y)$ met sp.

$f: X \rightarrow Y$ map. The followings are equiv.

1.) f is cont. 2.) $\forall A \subset X \ f(\bar{A}) \subset \overline{f(A)}$

Rem. $f(\bar{A}) \not\subset \overline{f(A)}$ means $\exists x \in X$ s.t.

• x can be approx. by points of A

• $f(x)$ cannot be approx. by points $f(a)$
for $a \in A$

Proof 1) $\Rightarrow$ 2) contrapos not 2) $\Rightarrow$ not 1)

Take x as above rem. so $\forall \varepsilon \exists a \in A$

$d(x, a) < \varepsilon$, $f(x)$ and $f(a)$ not close

(#) -- More formally $\exists \delta \forall \varepsilon \exists a \in A \ d(x, a) < \delta, d(f(x), f(a)) > \varepsilon$

This x violates continuity.

2) $\Rightarrow$ 1) again contrapos, not 1) $\Rightarrow$ not 2)

take $x \in X$ s.t. $\exists \delta \forall \varepsilon \exists \hat{y} \cdot d(x, \hat{y}) < \varepsilon$ and $d(f(x), f(\hat{y})) > \delta$

for such δ , for each ε choose y_ε as above

set $A = \{y_\varepsilon : \varepsilon > 0\}$

• $x \in \overline{A}$ because $\forall \varepsilon \ d(x, y_\varepsilon) < \varepsilon$

• $f(x) \notin \overline{f(A)}$ because $d(f(x), f(y_\varepsilon)) > \delta$

Thm 18.3 $(X, d_X), (Y, d_Y)$ met sp. suppose

1. $X = A \cup B$ A, B closed

2. $f: A \rightarrow Y$, $g: B \rightarrow Y$ cont

3. $\forall x \in A \cap B \quad f(x) = g(x)$

Then we have a (unique) cont. map $h: X \rightarrow Y$

s.t. $h(x) = f(x)$ for $x \in A$ $h(x) = g(x)$ for $x \in B$

Rem. we can repl. by " A, B open"

Proof Key point $\forall C \subset X \quad \overline{C} = \overline{C \cap A} \cup \overline{C \cap B}$



continuity of $f \Rightarrow f(\overline{C \cap A}) \subset \overline{f(C \cap A)}$

same for $g, C \cap B$ (can be replaced by h)

then $h(\overline{C}) \stackrel{\text{k.p.}}{=} h(\overline{C \cap A} \cup \overline{C \cap B})$

$= h(\overline{C \cap A}) \cup h(\overline{C \cap B}) \subset \overline{h(C \cap A)} \cup \overline{h(C \cap B)}$

for $x \in \overline{C} \subset \overline{h(C)}$

Proof of key point (c): for each $n \in \mathbb{N}$

take $y_n \in C$ s.t. $d(x, y_n) < 2^{-n}$

set $I = \{n : y_n \in C \cap A\}$, $J = \{n : y_n \in C \cap B\}$

one of these is infinite. if $|I| = \infty$ then $x \in \overline{C \cap A}$

§ 12 topological spaces

Sometimes metric spaces are not enough to understand approximations.

- operations on functions.: X some space.

$T: X \rightarrow X$ transform.

$$\varphi_n(f)(x) = \frac{1}{n} \sum_{0 \leq i < n} f(T^i x). \quad (f: X \rightarrow \mathbb{R})$$

$\leadsto$ do φ_n 's converge to some functional ψ ?

- algebraic geom.: only care about "subsets defined by polynomial equations"

Def. X : set. A topology is given by a collection $\mathcal{T}$ of subsets of X ($\mathcal{T} \subset \mathcal{P}(X) = 2^X$) such that

- 1) for any subcoll. $\mathcal{S} \subset \mathcal{T}: (\bigcup_{A \in \mathcal{S}} A) \in \mathcal{T}$
- 2) for any finite subcoll. $\mathcal{S} \subset \mathcal{T}: (\bigcap_{A \in \mathcal{S}} A) \in \mathcal{T}$
- 0). if $\mathcal{S} = \emptyset$ (empty collection) we interpret 1) as $\emptyset \in \mathcal{T}$, 2) as $X \in \mathcal{T}$.

We say $(X, \mathcal{T})$ is a topological space

$U \in \mathcal{T}$ is an open set for this topology

Examples

- $\mathcal{T} = \{A: A \subset X\} = \mathcal{P}(X)$. discrete topology on X .

- $X = \text{Map}(Y, \mathbb{R})$ for some set Y

$$\mathcal{T} = \left\{ A : \forall f \in A \exists y_1, \dots, y_n \in Y, \varepsilon > 0 \text{ s.t. } |f(y_i) - g(y_i)| < \varepsilon \ (i=1, \dots, n) \Rightarrow g \in A \right\}$$

topology of pointwise convergence

- $X = \mathbb{R}^n$ $\mathcal{T} = \{A: \exists \text{ set of polynomials } S \text{ (in } n \text{ variables).}$

s.t. $A = \{x = (x_1, \dots, x_n) : \exists f \in S, f(x) = 0\}$

