

§12 cont.

Comparison of topology

sometimes there are different (sensible) choices of topology on the same set.

Ex. $X = \mathbb{R}^n$

- topology from metric.
- top. defined by polynomials (last time) (Zariski topology).

Def. τ, τ' topologies on a set X .

τ' is finer than τ , or τ is coarser than τ' if $\tau' \supset \tau$ as collections of subsets

Ex. metric top. on $\mathbb{R}^n$ is finer than the Zariski top.

§13. Basis for topology

We often specify only "important" open sets we are interested in. (then use the axioms of topology to work out general ones).

Def. X set.

a basis of topology is a coll. B of subsets of X s.t.

$$(1). \forall x \in X \exists A \in B \text{ s.t. } x \in A$$

$$(2). \forall x \in X, A_1, A_2 \in B : x \in A_1 \cap A_2 \Rightarrow \exists A_3 \in B \text{ s.t. } x \in A_3, A_3 \subset A_1 \cap A_2$$

Examples $X = \mathbb{R}^2$

$$1. B = \{ B_d(x, \varepsilon) : x \in \mathbb{R}^2, \varepsilon > 0 \} \quad d: \text{usual met.}$$

checking (1): for any x , $x \in B_d(x, \varepsilon)$.

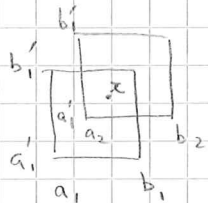
(2). if $x \in B_d(y, \varepsilon) \cap B_d(y', \varepsilon')$

take ε'' s.t. $B_d(x, \varepsilon'') \subset \text{both}$.



then $A_3 = B_d(x, \varepsilon'')$ will do.

$$2. \quad B = \{ \underbrace{(a, b)}_{\text{opn. int.}} \times \underbrace{(a', b')}_{\text{opn. int.}} : a, b, a', b' \in \mathbb{R} \}$$



Prop-Def. B : basis for a topology on X .

$$\mathcal{T} = \{ U \subset X : \forall x \in U \exists A \in B : x \in A, A \subset U \}$$

is a topology on X , called the top. generated by B .

Proof. We need to check

1. $\mathcal{S} \subset \mathcal{T}$ subcoll $\Rightarrow$ union $Y = \bigcup_{V \in \mathcal{S}} V$ is in $\mathcal{T}$

2. $\mathcal{S} \subset \mathcal{T}$ finite subcoll $\Rightarrow$ intersection $Z = \bigcap_{V \in \mathcal{S}} V \in \mathcal{T}$.

Proof of 1. if $\mathcal{S} = \emptyset$ then $Y = \emptyset$, formally $\emptyset \in \mathcal{T}$.

otherwise we want to check $\forall x \in Y \exists A \in B : x \in A \subset Y$

fix x ; take $V \in \mathcal{S}$ s.t. $x \in V$.

then $\exists A \in B$ s.t. $x \in A \subset V$ this A will do

Proof of 2. again we want $\forall x \in Z \exists A \in B : x \in A \subset Z$

fix x ; for each $V \in \mathcal{S}$ take $A_V \in B$ s.t.

$x \in A_V \subset V$ (possible by $x \in V$)

by induction $\exists A \in B : x \in A \subset A_{V_1} \cap \dots \cap A_{V_n}$

($\mathcal{S} = \{V_1, \dots, V_n\}$)

this A will do.

Lem 13.1 $\mathcal{B}$: basis for a top. on X .

$\mathcal{T}$: top. gen'ed by $\mathcal{B}$.

then $\mathcal{T} = \{U \subset X : U = \bigcup_{A \in \mathcal{S}} A \text{ for some } \mathcal{S} \subset \mathcal{B}\}$

Sketch: $\mathcal{T} \supset \text{RHS}$ is from cond. (2) for top. & $\mathcal{B} \subset \mathcal{T}$

$\mathcal{T} \subset \text{RHS}$: Suppose $U \in \mathcal{T}$

then $U = \bigcup_{x \in U} A_x$ for $x \in A_x \in \mathcal{B}$.

Lem 13.3. X : set. $\mathcal{B}, \mathcal{B}'$ bases for top.

$\mathcal{T}$: top. gen'ed by $\mathcal{B}$, $\mathcal{T}'$: top. - by $\mathcal{B}'$.

The followings are equiv.

1) $\mathcal{T}'$ is finer than $\mathcal{T}$ ($\mathcal{T}' \supset \mathcal{T}$)

2) $\forall x \in X, A \in \mathcal{B}$ s.t. $x \in A \exists A' \in \mathcal{B}'$ s.t. $x \in A' \subset A$.

Proof $2 \Rightarrow 1$. fix $U \in \mathcal{T}$ want $U \in \mathcal{T}'$ i.e.

$\forall x \in U \exists A' \in \mathcal{B}' : x \in A' \subset U$.

take $x \in U$. $U \in \mathcal{T}$ implies $\exists A \in \mathcal{B}$ $x \in A \subset U$

2) gives A' we wanted

1. $\Rightarrow$ 2. take $x \in X, A \in \mathcal{B}$ s.t. $x \in A$

want $A' \in \mathcal{B}'$ s.t. $x \in A' \subset A$

$\mathcal{B} \subset \mathcal{T} \subset \mathcal{T}' \Rightarrow A \in \mathcal{T}'$ this means
 $\mathcal{B} \xrightarrow{\text{sets } \mathcal{T}} \mathcal{T}' \text{ finer than } \mathcal{T}$

$\exists A' \in \mathcal{B}'$ as we wanted $\square$

Application $X = \mathbb{R}^2$.

$\mathcal{B} = \{B_d(x, \epsilon) : x \in \mathbb{R}^2, \epsilon > 0\}$

$\mathcal{B}' = \{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$

generate same top.



(X, T) top. sp.

Def. an (open) neighborhood of $x \in X$ is
 an open set U s.t. $x \in U$.

another convention: a neighborhood of $x \in X$:

a subset $A \subset X$ s.t. $\exists U \subset X$ open, $x \in U \subset A$

Neighborhood of x : a range "close enough" to x