

§15 Product topology.

(cartesian) product set $X \times Y$: the set of
 $(X, Y: \text{sets}) \rightarrow$ ordered pairs (x, y) $x \in X$
 $y \in Y$.
 (even if $X=Y$ dist. (x, x') & (x', x)).

topology on X and $Y \rightarrow$ we want to make
 sense of topology so " (x_1, y_1) close to (x_2, y_2) "
 means " x_1 close to x_2 " and " y_1 close to y_2 "

Def basis for product topology:

$$\mathcal{B} = \{U \times V : U \text{ open set of } X, V \text{ open set of } Y\}$$

Checking the conditions for basis (Aus 29)

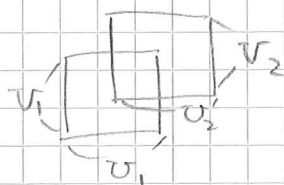
$$1. \quad \forall (x, y) \in X \times Y \quad \exists A \in \mathcal{B} : (x, y) \in A$$

$$2. \quad \forall (x, y) \in X \times Y, A_1, A_2 \in \mathcal{B} \text{ s.t. } (x, y) \in A_1 \cap A_2 \\ \exists A_3 \in \mathcal{B} \text{ s.t. } (x, y) \in A_3 \subset A_1 \cap A_2$$

$$1. \quad \text{take } U, V \text{ s.t. } x \in U \text{ and } y \in V \\ \hookrightarrow \text{can be } X, Y$$

$$2. \quad \text{write } A_1 = U_1 \times V_1, A_2 = U_2 \times V_2$$

$$(x, y) \in A_1 \cap A_2 \text{ means } x \in U_1 \cap U_2 \text{ and } y \in V_1 \cap V_2$$



$$A_3 = (U_1 \cap U_2) \times (V_1 \cap V_2) \\ \uparrow \quad \uparrow \\ \text{open} \quad \text{open}$$

Thm 15.1 X, Y top. sp., $\mathcal{B}, \mathcal{C}$: bases for
 their top.; the collection

$\mathcal{D} = \{E \times F : E \in \mathcal{B}, F \in \mathcal{C}\}$ is a basis for
 the prod. top. on $X \times Y$.

We will use: Lem 13.2 Z : top. sp., $\mathcal{A}$: coll. of
 open sets of Z ; $\mathcal{A}$ is a basis for the top. of Z

$$\Leftrightarrow \forall z \in Z, U \subset Z \text{ open } \exists V \in \mathcal{A} \text{ s.t. } z \in V \subset U.$$

Proof of Thm ; check that $\mathcal{D}$ satisfies the cond. of Lem. i.e. $\forall (x,y), W \subset X \times Y$ s.t. $(x,y) \in W$ open
 $\exists E \times F \in \mathcal{D}$ s.t. $(x,y) \in E \times F \subset W$.

How does W look like? (Aug 29)

$$W \subset X \times Y \text{ open} \Leftrightarrow \forall (x',y') \in W \exists T \in \mathcal{D} : (x',y') \in T \subset W$$

then $T = E \times F$ is what we are looking for.

Proof of Lem ; $\Rightarrow$ is by def ; we check $\Leftarrow$:

$$\text{St 1 } \forall z \in Z \exists V \in \mathcal{A} : z \in V \quad \text{flows from } Z = \bigcup$$

$$\text{St 2 } \forall z \in Z, V_1, V_2 \in \mathcal{A} \text{ s.t. } z \in V_1 \cap V_2$$

$$\exists V_3 \in \mathcal{A} : z \in V_3 \subset V_1 \cap V_2$$

use the cond. for $V_1 \cap V_2 = \bigcup$

Next: this $\mathcal{A}$ recovers the top. of Z .

$$\text{St 3 } \text{top defined by } \mathcal{A} \text{ is coarser than the original one} \\ (\mathcal{T}') \quad (\mathcal{T})$$

" $\bigcup$ open for $\mathcal{T}'$ " means

$$\forall z \in \bigcup \exists V \in \mathcal{A} : z \in V \subset \bigcup ; \text{ choose } V_z$$

$$V_z \text{ is open for } \mathcal{A} \Rightarrow \bigcup = \bigcup_{z \in \bigcup} V_z \text{ is open for } \mathcal{T}$$

$$\text{St 4 } \mathcal{T}' \text{ is finer than } \mathcal{T}$$

use the cond. in claim, argue as St. 3. $\square$

$$\text{Example. } X = Y = \mathbb{R}, \quad \mathcal{B} = \mathcal{C} = \{ (a,b) : a,b \in \mathbb{R} \} \text{ open int}$$

$$\text{so } \mathcal{D} = \{ (a,b) \times (c,d) : a,b,c,d \in \mathbb{R} \}$$

$\mathcal{D}$ generates the prod. top. of $\mathbb{R}^2$.

we also know that $\mathcal{D}$ gen's the metric top. of $\mathbb{R}^2$ (Aug. 29)

Forget $X \times Y$ is the categorical direct product

$$\{ \text{cont map } f : Z \rightarrow X \times Y \} \cong \{ (g,h) : \text{cont } g : Z \rightarrow X, h : Z \rightarrow Y \}$$

§ 16 Subspace topology

Prop-Def. $(X, \mathcal{T})$ top. sp. $Y \subset X$ subset. Then

$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is a topology on Y .

We call this the subspace topology (or induced topology)

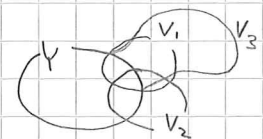
Proof. 1. union for subcoll $S \subset \mathcal{T}_Y$; $\bigcup_{V \in S} V \stackrel{?}{\in} \mathcal{T}_Y$.

consider. $S' = \{U \in \mathcal{T} : Y \cap U \in S\}$

then $S = \{Y \cap U : U \in S'\}$ (with a lot of redundancy).

Put $W' = \bigcup_{V' \in S'} V'$. this is open in X .

Observe $W = Y \cap W'$



2. intersection for finite subcoll $S \subset \mathcal{T}_Y$.

$W = \bigcap_{V \in S} V \stackrel{?}{\in} \mathcal{T}_Y$. $S = \{V_1, \dots, V_n\}$

for each i , fix one $V'_i \in \mathcal{T}$ s.t. $V_i = Y \cap V'_i$

$\Rightarrow \begin{cases} \text{then } W' = V'_1 \cap \dots \cap V'_n \in \mathcal{T} \\ W = Y \cap W' \end{cases}$ $\square$

Caution. $Z \subset Y$ can be open for $\mathcal{T}_Y$ but not open for $\mathcal{T}$ ~~$(-1, 1)$~~

Ex. $X = \mathbb{R}$, $Y = [0, \infty)$. $Z = [0, 1) = Y \cap (-1, 1)$

