

## §17 Closed sets, limit points.

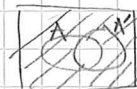
Def.  $(X, \tau)$  top. sp.a closed set of  $X$  is a subset  $A \subset X$  s.t. $U = X \setminus A = \{x \in X : x \notin A\}$  is open. ( $U \in \tau$ ).

Th'm (17.1)

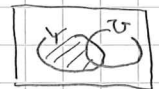
1.  $S$  : some coll. of closed sets of  $X$  $\Rightarrow B = \bigcap_{A \in S} A$  is closed2.  $S$  : some finite coll. of closed sets. $\Rightarrow B = \bigcup_{A \in S} A$  is closed.

Proof. Use De Morgan's law

$$X \setminus \bigcap_{A \in S} A = \bigcup_{A \in S} X \setminus A$$



Th'm (17.2)

 $A \subset Y \subset X$  $A$  closed for the subsp. top. of  $Y$ . $\Leftrightarrow A = Y \cap B$ ,  $B$  closed set of  $X$ Recall subsp. top.  $\tau_Y = \{Y \cap U : U \in \tau\}$ Proof.  $Y \setminus (Y \cap U) = Y \cap (X \setminus U)$   
open for  $\tau_Y$ Def  $A \subset X$ • the interior of  $A$  :  $\text{Int } A = \bigcup_{U \text{ open}, U \subset A} U$  (also  $\overset{\circ}{A}$ ).• the closure of  $A$  :  $\bar{A} = \bigcap_{B \text{ closed}, A \subset B} B$  (also  $\text{cl}(A)$ ).Rem  $X \setminus \bar{A} = \text{Int}(X \setminus A)$ ,  $\text{Int } A$  open,  $\bar{A}$  closed.Th'm (17.5).  $A \subset X$ a)  $x \in \bar{A} \Leftrightarrow \forall U \text{ open} : x \in U \Rightarrow U \cap A \neq \emptyset$ .any neighborhood of  $x$  intersects with  $A$ .b) Suppose  $\mathcal{B}$  is a basis of the top.  $\tau$ . Then $x \in \bar{A} \Leftrightarrow \forall U \in \mathcal{B} : x \in U \Rightarrow U \cap A \neq \emptyset$

Proof a) We prove contrapos. ; want

$$x \in X \setminus \bar{A} \Leftrightarrow \exists U \text{ open : } x \in U \text{ and } U \cap A = \emptyset$$

$$\Rightarrow : \bar{A} \text{ closed} \Rightarrow U = X \setminus \bar{A} \text{ open}$$

$$\Leftarrow : \text{take such } U, \text{ then } B = X \setminus U \text{ is}$$

$$\underbrace{\text{closed}, A \subset B, x \notin B}_{\Leftrightarrow \bar{A} \subset B} \quad \Leftrightarrow x \notin \bar{A}$$

$$b) \text{ reduces to a) via } U : \text{open, } x \in U \Rightarrow \exists V \in \mathcal{B} \quad x \in V \subset U.$$

Examples.  $X = \mathbb{R}^n$

$$1. \quad T : \text{met. top.} \quad x \in \bar{A} \stackrel{(b)}{\Leftrightarrow} \forall V = B_d(y, \varepsilon) \{x \in V \Rightarrow V \cap A \neq \emptyset\}$$

$$\Leftrightarrow \forall \varepsilon, y \{d(x, y) < \varepsilon \Rightarrow \exists a \in A : d(a, y) < \varepsilon\}$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists a \in A \quad d(x, a) < \varepsilon.$$

$$2. \quad T = \{U : \exists \text{ set of polynoms } S \text{ s.t. } U = \{x = (x_1, \dots, x_n) : \exists f \in S \quad f(x) \neq 0\}\}$$

(Zariski topology, Aug 25)

$$x \in \bar{A} \Leftrightarrow \forall \text{ polynom } f. [f|_A = 0 \Rightarrow f(x) = 0]$$

Main point : closed sets of  $T$  are

$$V(S) = \{x \in \mathbb{R}^n : \forall f \in S : f(x) = 0\} \quad (S \subset \{\text{polynoms}\})$$

$$\text{So. } \bigcap B_d(x, \varepsilon) = \mathbb{R}^n \quad \left( \begin{array}{l} f(y) = 0 \text{ for all } d(x, y) < \varepsilon \\ \text{poly.} \end{array} \Rightarrow f = 0 \right)$$

Def.  $A \subset X$  ; a limit point (or accumulation point, cluster point) of  $A$  is a point  $x \in X$  s.t.:

$$U \text{ open, } x \in U \Rightarrow \exists y \neq x, y \in A \cap U.$$

$$\text{Rem } x \text{ lim. pt of } A \Leftrightarrow x \in \overline{A \setminus \{x\}}$$

$$\text{Ex. } A = \left\{ \frac{1}{n} : n = 1, 2, \dots \right\} \subset \mathbb{R} \quad 0 \text{ lim pt, } \frac{1}{n} \text{ not.}$$

Thm (17.6)  $A \subset X$  ;  $A' = \{x : \text{lim. pt. of } A\}$ .

$$\text{Then } \bar{A} = A \cup A'$$

Proof  $A \cup A' \subset \bar{A}$  :  $A \subset \bar{A}$  is obvious.  $A' \subset \bar{A}$

$$x \in A' \Rightarrow x \in \overline{A \setminus \{x\}}, \quad \overline{A \setminus \{x\}} \subset \bar{A}$$

$\bar{A} \subset A \cup A'$  : enough to prove  $\bar{A} \setminus A \subset A'$

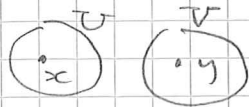
Take  $x \in \bar{A} \setminus A$  By  $x \in \bar{A}$  any open  $U$  s.t.

$x \in U$  intersects  $A$ , i.e.  $\exists y \in U \cap A$

$x \notin A \Rightarrow y \neq x$ . This shows  $x \in A'$   $\square$

Def.  $(X, \tau)$  is a Hausdorff space if

$\forall x \neq y \in X \exists$  open  $U, V$  s.t.  $x \in U, y \in V, U \cap V = \emptyset$



$(X, \tau)$  is a  $T_1$ -space if  $\forall x \in X, \{x\}$  is closed  
(i.e.  $X \setminus \{x\} \in \tau$ )

Rem. Hausdorff  $\Rightarrow T_1$  ;  $X \setminus \{x\} = \bigcup_{y \neq x} V$  as above.

Examples.

1. metric spaces are Hausdorff.

2.  $\mathbb{R}^n$  with Zariski top. is not Hausdorff, but  $T_1$

limit of sequence

$x_1, x_2, \dots$  sequence of points in  $X$

Def a limit of  $(x_n)_n$  is a point  $y$  s.t.

$\forall U$  open  $y \in U \exists n \forall N \geq n \ x_N \in U$   
(neighborhood of  $y$ )

Ex. • met sp. : usual limit.  $d(y, x_n) \rightarrow 0$

•  $x_n = \frac{1}{n}$  ( $n = 1, 2, \dots$ ) for  $X = \mathbb{R}$ , Zariski top.

$\Rightarrow$  any  $y \in \mathbb{R}$  is a lim. of  $(x_n)_n$

