

§18 Continuous functions

Basic idea: $f(x)$ is a continuous function if:

"when x moves a little, $f(x)$ moves only a little"

How to turn this to a formal def:

U : (open) neighborhood of $f(x)$ controlling

$\Rightarrow \exists V$: (open) neighborhood of x controlling

s.t. $y \in V \Rightarrow f(y) \in U$

More streamlined:

Def $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ top. sp., $f: X \rightarrow Y$ map

f is continuous if $\forall U \in \mathcal{T}_Y$ open $(U \in \mathcal{T}_Y)$

$f^{-1}(U) = \{x \in X : f(x) \in U\} \subset X$ is open ($f^{-1}(U) \in \mathcal{T}_X$)

Rem $U \in \mathcal{T}_Y$ is an open neigh. of $f(x)$ for

any $x \in f^{-1}(U) \rightsquigarrow$ finding V as above

for each $x \in f^{-1}(U)$ means $f^{-1}(U)$ is open

Prop. $\mathcal{B}$ basis for the top. of Y

$f: X \rightarrow Y$ is cont. $\Leftrightarrow \forall U \in \mathcal{B} : f^{-1}(U) \subset X$ is open

For metric spaces: " ε - δ -definition of continuity"

$(X, d_X), (Y, d_Y)$ met sp. f is cont. if

$\forall x \in X, \delta > 0 \exists \varepsilon > 0 : d_X(x, x') < \varepsilon \Rightarrow d_Y(f(x), f(x')) < \delta$

reduce to the above def. for maps of top. sp. by

$\mathcal{B} = \{B_{d_Y}(y, \delta) : y \in Y, \delta > 0\}$

$(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ top. sp., $f: X \rightarrow Y$ map

Def. $x \in X$; f is continuous at x if

$\forall U$: (open) neighborhood of $f(x)$:

$\exists V$: (open) neighborhood of x s.t. $[y \in V \Rightarrow f(y) \in U]$

Thm 18.1 same setting as above

The following are equivalent

1. f is continuous

2. $\forall A \subset X: f(\bar{A}) \subset \overline{f(A)}$

3. $\forall B \subset Y$ closed: $f^{-1}(B)$ closed in X

4. $\forall x \in X: f$ is cont. at x

Proof $1 \Rightarrow 2$ idea: $x \in \bar{A}$ means " x is approx'd by points $y \in A$ " continuity $f(y)$ approximate $f(x)$

Formally: want $x \in \bar{A} \Rightarrow \underbrace{f(x) \in \overline{f(A)}}_{\text{any neigh. of } f(x) \text{ intersects } f(A)}$

take an open neigh. U of $f(x)$

$\Rightarrow f^{-1}(U)$ open set in X , contains x
 f : cont

$\Rightarrow_{x \in \bar{A}} f^{-1}(U)$ intersects A i.e. $\exists y \in f^{-1}(U) \cap A$

$\Rightarrow f(y)$ is in $U \cap f(A)$

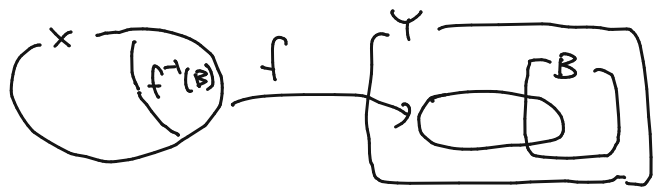
$2 \Rightarrow 3$ "closed" means "closed under approximation"

Fix a closed $B \subset Y$, set $A = f^{-1}(B)$. Want $A = \bar{A}$

2. gives $f(\bar{A}) \subset \overline{f(A)}$, while $f(A) \subset B$

$\Rightarrow f(\bar{A}) \subset \bar{B} = B$ so $\bar{A} \subset f^{-1}(B) = A$

3 $\Leftrightarrow$ 1 Consequence of $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$



4 $\Leftrightarrow$ 1 Unpacking def. of open sets, neighborhood, ... $\square$

Some formal rules (Thm 18.2)

1. $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ continuous maps
 $\Rightarrow g \circ f : X \rightarrow Z$ is continuous

2. $(U_\alpha)_{\alpha \in I}$ collection of open sets of X
 $f : X \rightarrow Y$ map.

$\forall \alpha$ restriction $f|_{U_\alpha} : U_\alpha \rightarrow Y$ continuous
 $\Leftrightarrow f$ is continuous

Proof. 1 : $(g \circ f)^{-1}(U) = \underbrace{f^{-1}(g^{-1}(U))}_{\substack{\text{Open by continuity of } f \\ \Leftarrow \text{Open by continuity of } g}} ; \text{ if } U \text{ open}$

2. we check continuity at x for all $x \in X$

Suppose $x \in U_\alpha$

V : open neighborhood of x in X

$\Rightarrow V \cap U_\alpha$: open neighborhood of x in U_α

W : open neighborhood of x in U_α

$\Rightarrow W$: open neighborhood of x in X

these allow us to say

f is cont. at $x \Leftrightarrow f|_{U_\alpha}$ is cont. at x $\square$

Def. $(X, T_X), (Y, T_Y)$ top. sp. A homeomorphism from X to Y is a map $f: X \rightarrow Y$

- | | | |
|---|---|--|
| <ol style="list-style-type: none"> 1. f is bijective 2. f is continuous 3. f^{-1} is continuous | } | f gives an identification of X & Y as top. sp. |
|---|---|--|

We say X and Y are homeomorphic if such f exists.

An imbedding (or embedding) of X into Y is an injective map $f: X \rightarrow Y$ that is a homeo. as a map $X \rightarrow f(X)$
 $\quad \quad \quad \nwarrow$ with induced topology

Examples

- $S^1 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$ and $E = \{ (x, y) \in \mathbb{R}^2 : \frac{1}{2}x^2 + y^2 = 1 \}$ are homeo.



- $f: \mathbb{R} \rightarrow \mathbb{R}^2, x \mapsto (x, 0)$ is an embedding