

§ Product topology

Goal: make sense of topology on infinite product of top. sp.

Motivation infinite product of $X = \{0, 1\}$.

$$X \times X \times \dots = \{ (x_i)_{i=0}^{\infty} : x_i = 0, 1 \}$$

- 2-adic integer $\sum_{i=0}^{\infty} x_i \cdot 2^i = x_0 \cdot 1 + x_1 \cdot 2 + x_2 \cdot 4 + \dots$

- Cantor set $[0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{0 \leq k < 3^n} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$

lim. of $\frac{1}{3^n}$ $\rightarrow$ $\frac{0}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3}$ $\rightarrow$ $\frac{0}{3} \frac{1}{3} \frac{1}{3}$
 $x_0=0$ $x_0=1$ $x_1=0$ $x_1=1$ $x_2=0$ $x_2=1$

Sensible topology: $x = (x_i)_i$ and $y = (y_i)_i$ are close if $x_i = y_i$ for $i \leq N$.
 N bigger $\leftrightarrow$ x and y closer.

More formal setting: family of top. sp. $(X_\alpha, \tau_\alpha)_{\alpha \in I}$.

direct product as set: $\prod_{\alpha \in I} X_\alpha = \{ (x_\alpha)_{\alpha \in I} : x_\alpha \in X_\alpha \}$.

(also write $\prod X_\alpha$, etc.) formally the set of maps

$$f: I \rightarrow \bigcup_{\alpha \in I} X_\alpha \quad \text{s.t.} \quad f(\alpha) \in X_\alpha$$

projection map: $\pi_\beta: \prod_{\alpha} X_\alpha \rightarrow X_\beta, (x_\alpha)_\alpha \mapsto x_\beta$
 for each $\beta \in I$.

Prop-Def. The collection of subsets of $\prod_{\alpha} X_\alpha$ defined as

$$\mathcal{B} = \{ \underbrace{\pi_{\beta_1}^{-1}(U_1) \cap \dots \cap \pi_{\beta_k}^{-1}(U_k)}_{U_i \subset X_{\beta_i} \text{ open } (U_i \in \tau_{\beta_i})} : k=0, 1, \dots, \beta_i \in I \}$$

is a basis for a top. on $\prod_{\alpha} X_\alpha$. This top. is called the product topology

$$\rightarrow \{ (x_\alpha)_\alpha : x_\alpha \in U_{\beta_i} \text{ if } \alpha = \beta_i, i=1, \dots, k \}$$

Proof of claim: we need to check (Aug 29).

(1). $\forall x \in \prod X_\alpha \quad \exists A \in \mathcal{B} : x \in A$

(2). $\forall x \in \prod X_\alpha, A_1, A_2 \in \mathcal{B} : [x \in A_1 \cap A_2 \Rightarrow \exists A_3 \in \mathcal{B} : x \in A_3 \subset A_1 \cap A_2]$

1) we can take $k=0$, that gives $\prod_{\alpha} X_{\alpha}$ itself.

(or use any k , any β_i , choose $U_i \ni x_{\beta_i}$)

2) suppose $x \in A_1 \cap A_2$. write

$$A_1 = \pi_{\beta_1}^{-1}(U_1) \cap \dots \cap \pi_{\beta_k}^{-1}(U_k)$$

$$A_2 = \pi_{\gamma_1}^{-1}(V_1) \cap \dots \cap \pi_{\gamma_j}^{-1}(V_j)$$

$$\text{Then } A_3 = A_1 \cap A_2 = \pi_{\delta_1}^{-1}(W_1) \cap \dots \cap \pi_{\delta_{k+j}}^{-1}(W_{k+j}) \in \mathcal{B}.$$

$$\delta_i = \beta_i, W_i = U_i \text{ for } i \leq k, \quad \delta_i = \gamma_{i-k}, W_i = V_{i-k} \text{ for } i > k$$

Rem. even if each $(X_{\alpha}, \tau_{\alpha})$ is discrete
($\forall A \subset X_{\alpha}$ is in τ_{α})

$\prod_{\alpha} X_{\alpha}$ might not be discrete.

Ex. $I = \{0, 1, \dots\}$, $X_{\alpha} = \{0, 1\}$ for each $\alpha \in I$

$\Rightarrow \prod_{\alpha} X_{\alpha}$ is homeo. to the Cantor set with induced top. as a subset of $\mathbb{R}$.

Another candidate: use $\mathcal{B} = \left\{ \prod_{\alpha} U_{\alpha} : U_{\alpha} \subset X_{\alpha} \text{ open} \right\}$

$\Rightarrow$ box topology on $\prod_{\alpha} X_{\alpha}$.

$|I| < \infty \Rightarrow$ prod. top. = box top.

but prod. top is more sensible if $|I| = \infty$.

Thm 19.5 $(X_{\alpha}, \tau_{\alpha})_{\alpha \in I}$ top sps., $A_{\alpha} \subset X_{\alpha}$

Then $\overline{\prod_{\alpha} A_{\alpha}} = \prod_{\alpha} \overline{A_{\alpha}}$ for prod. (and box) top.

Proof St. 1. $\prod_{\alpha} \overline{A_{\alpha}} \subset \overline{\prod_{\alpha} A_{\alpha}}$: take $x \in \prod_{\alpha} \overline{A_{\alpha}}$.

want: $x \in \overline{\prod_{\alpha} A_{\alpha}}$ i.e. $\forall$ open neigh. U of x : $U \cap \prod_{\alpha} A_{\alpha} \neq \emptyset$

• we can assume $U = \pi_{\beta_1}^{-1}(U_1) \cap \dots \cap \pi_{\beta_k}^{-1}(U_k)$, $\beta_i \neq \beta_j$
($x \in U \Leftrightarrow x_{\beta_i} \in U_i$ for $i=1, \dots, k$)

for each i , $x_{\beta_i} \in \overline{A_{\beta_i}} \Rightarrow U_i \cap A_{\beta_i} \neq \emptyset$. i.e. $\exists y_i \in U_i \cap A_{\beta_i}$

• define $z \in \prod_{\alpha} A_{\alpha}$ by: $z_{\beta_i} = y_i$, choose

whatever $z_{\alpha} \in A_{\alpha}$ for $\alpha \neq \beta_1, \dots, \beta_k$

$\uparrow$ nonempty; otherwise $\prod_{\alpha} A_{\alpha} = \emptyset$

• then $z \in U$ by $\pi_{\beta_i}(z) = y_i \in U_i$

St. 2 $\overline{\prod_{\alpha} A_{\alpha}} \subset \prod_{\alpha} \overline{A_{\alpha}}$: take $x \in \overline{\prod_{\alpha} A_{\alpha}}$, want $x_{\alpha} \in \overline{A_{\alpha}}$

i.e. $\forall$ open neigh. U of x_{α} $U \cap A_{\alpha} \neq \emptyset$

cont.) Set $V = \pi_\alpha^{-1}(U)$ this is open in $\prod_\alpha X_\alpha$
 and $x \in V \Rightarrow V \cap \pi_\alpha A_\alpha \neq \emptyset$

take $y \in V \cap \pi_\alpha A_\alpha$ then $y_\alpha \in U \cap A_\alpha$. $\square$

Thm 19.6 $(X_\alpha, \tau_\alpha)_{\alpha \in I}$ top. sps., (Y, τ') top. sp.

$f_\alpha: Y \rightarrow X_\alpha$ map $(\alpha \in I)$

Put $f: Y \rightarrow \prod_\alpha X_\alpha$, $y \mapsto (f_\alpha(y))_{\alpha \in I}$.

Then f is cont. (for the prod. top. on $\prod_\alpha X_\alpha$)

$\Leftrightarrow \forall \alpha \in I$ f_α is cont.

Proof. $\Rightarrow$: $f_\alpha = \pi_\alpha \circ f$, π_α proj. map., compos. of cont.

maps are cont. (or directly check $f_\alpha^{-1}(U) = f^{-1}(\pi_\alpha^{-1}(U))$.)

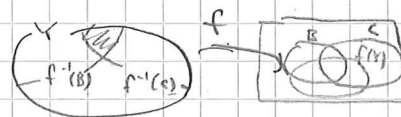
$\Leftarrow$: $B = \{ \pi_{\beta_1}^{-1}(U_1) \cap \dots \cap \pi_{\beta_k}^{-1}(U_k) : U_i \subset X_{\beta_i} \text{ open} \}$

was a basis for the prod. top. $\Rightarrow$ enough to check (sep. 15)

$\forall A \in B$: $f^{-1}(A) \subset Y$ open.

Write $A = \pi_{\beta_1}^{-1}(U_1) \cap \dots \cap \pi_{\beta_k}^{-1}(U_k)$

$\bullet f^{-1}(A) = \bigcap_{i=1}^k f^{-1}(\pi_{\beta_i}^{-1}(U_i))$ ($f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$)



$\bullet f^{-1}(\pi_{\beta_i}^{-1}(U_i)) = f_{\beta_i}^{-1}(U_i) \subset Y$ is open
 continuity of f_{β_i}

$\bullet \bigcap_{i=1}^k f^{-1}(\pi_{\beta_i}^{-1}(U_i)) \subset Y$ is open

finite intersection of open sets is open. $\square$

Rem. $I = Y$, $X_\alpha = X$ for fixed (X, τ)

prod. top. on $\prod_{y \in Y} X = \text{Map}(Y, X)$

$=$ top. of pointwise convergence.

