

§ 20 (reprise), 21 Metric spaces, continued

Recall: metric space (X, d)

$$d(y, x) = d(x, y) \geq 0, \quad d(x, y) + d(y, z) \geq d(x, z)$$

basis of top.: $\mathcal{B} = \{B_d(x, \varepsilon) = \{y : d(x, y) < \varepsilon\} : x \in X, \varepsilon > 0\}$

We know:

 $\mathbb{R}^2$ with Euclidean met $\underset{\text{homeo.}}{\sim} \mathbb{R}^2$ with prod. top.

basis as above.

basis $\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}\}$ abstractly: the prod. top. on $\mathbb{R}^2$ is metrizablei.e. $\exists$ met. d realizing this top.

Generalization to sequence of metric spaces.

Thm (cf. 20.5). (X_i, d_i) ($i = 1, 2, \dots$) met sp.1. $\prod_{i=1}^{\infty} X_i$ has a metric $D(x, y) = \sup_{i=1, 2, \dots} \frac{\bar{d}_i(x_i, y_i)}{i}$ where $\bar{d}_i(x_i, y_i) = \min(d_i(x_i, y_i), 1)$ 2. the top. given by D is the prod. top.Proof. 1. Step 1 D is well-defined ($D(x, y) \in \mathbb{R}$)we have $0 \leq D(x, y) \leq 1$ Step 2. D is a metric.Key point: $D(x, y) + D(y, z) \geq D(x, z)$ For each i , we have $\bar{d}_i(x_i, y_i) + \bar{d}_i(y_i, z_i) \geq \bar{d}_i(x_i, z_i)$ $\Rightarrow$ same rel after $\sup_i$.

2. comp. of tops: compare bases

$$\mathcal{B}_p = \left\{ B_{\bar{d}_1}(x_1, \varepsilon_1) \times \dots \times B_{\bar{d}_k}(x_k, \varepsilon_k) \times X_{k+1} \times X_{k+2} \times \dots : \right. \\ \left. k = 0, 1, \dots, x_i \in X_i, \varepsilon_i > 0 \right\}$$

$$\mathcal{B}_D = \{ B_D(x, \varepsilon) : x \in \prod_i X_i, \varepsilon > 0 \}$$

Step 1. B_P generates the prod. top.

- $\bar{d}_i$ def's the top. of (X_i, d_i)

$$B_{\bar{d}_i}(x, \varepsilon) = B_{d_i}(x, \varepsilon) \quad \text{for } \varepsilon \leq 1.$$

- $B_i (= \{ B_{\bar{d}_i}(x_i, \varepsilon) : x_i \in X_i, \varepsilon > 0 \})$ is a basis for X_i

$\Rightarrow$ claim.

Prod. top. has basis $B'_P = \{ \pi_{i_1}^{-1}(U_1) \cap \dots \cap \pi_{i_k}^{-1}(U_k) : k=0,1,\dots, U_j \in X_{i_j} \text{ open} \}$.

we may assume $i_j \neq i_{j'}$ for $j \neq j'$. then

$$\pi_{i_1}^{-1}(U_1) \cap \dots \cap \pi_{i_k}^{-1}(U_k) = Y_1 \times Y_2 \times \dots$$

$$Y_{i_j} = U_j, \quad Y_{i_j} \subset X_{i_j} \text{ if } i_j \neq i_{j'}$$

$\Rightarrow$ So $B_P \subset B'_P$, and $\forall x \in A' \in B'_P \exists A \in B_P : x \in A \subset A'$

Step 2. prod. top. is finer than the D-top.

$$\forall x \in A' \in B_D \exists A \in B_P : x \in A \subset A'$$

- Write $A' = B_D(x', \varepsilon')$

we may assume $A' = B_D(x, \varepsilon)$



- find N s.t. $A = B_{\bar{d}_1}(x_1, \varepsilon) \times \dots \times B_{\bar{d}_N}(x_N, \varepsilon) \times X_{N+1} \times \dots$

satisfies $A \subset A'$; want $y \in A \Rightarrow D(x, y) < \varepsilon$.

choose $N > \frac{1}{\varepsilon}$ then $y \in A$ means

$$\bar{d}_i(x_i, y_i) < \varepsilon \quad \text{for } i=1, \dots, N.$$

$$\text{then } D(x, y) \leq \sup \left\{ \underbrace{\frac{\bar{d}_1(x_1, y_1)}{1}, \dots, \frac{\bar{d}_N(x_N, y_N)}{N}}_{\text{dom. by } \varepsilon}, \underbrace{\frac{1}{N+1}, \dots}_{\bar{d}_i \leq 1} \right\}$$

$< \varepsilon$.

Step 3. prod. top. is coarser than the D-top.

$$\forall x \in A \in B_P \exists A' \in B_D : x \in A' \subset A$$

- Write $A = B_{\bar{d}_1}(y_1, \varepsilon_1) \times B_{\bar{d}_2}(y_2, \varepsilon_2) \times \dots \times B_{\bar{d}_k}(y_k, \varepsilon_k) \times X_{k+1} \times \dots$

we may assume $y_i = x_i$



- Find ε s.t. $A' = B_D(x, \varepsilon)$ satisfies $A' \subset A$.

$$\text{want } D(x, z) < \varepsilon \Rightarrow \bar{d}_i(x_i, z_i) < \varepsilon_i$$

$$\text{cont.}) \quad \varepsilon = \min \left\{ \frac{\varepsilon_i}{i} : i = 1, \dots, k \right\}$$

$$\text{if } D(x, z) < \varepsilon \text{ then } \frac{\overline{d_i}(x_i, z_i)}{i} \leq D(x, z) < \varepsilon$$

$$\Rightarrow \overline{d_i}(x_i, z_i) < \varepsilon_i \quad \square$$

Prop. (X, d) met. sp. $\Rightarrow$ Hausdorff as top. sp.

Thm (21.3). $(X, d_X), (Y, d_Y)$ met. sps.

$f: X \rightarrow Y$ map.

Then f cont $\Leftrightarrow \forall$ seq. $(x_n)_n$ in X s.t. $x_n \rightarrow x$
 $f(x_n) \rightarrow f(x)$

Proof $\Rightarrow$: follows from unpacking defs. (Seq. 15)

$\Leftarrow$: we check $\forall A \subset X \quad f(\overline{A}) \subset \overline{f(A)}$

$x \in \overline{A}$ means $\exists$ seq. $(x_n)_n \subset A$ s.t. $x_n \rightarrow x$
 (Lem 21.2: $B_d(x, \varepsilon) \cap A \neq \emptyset$)

Assumption $\Rightarrow f(x_n) \rightarrow f(x)$

So we get $f(x) \in \overline{f(A)} \quad \square$

