

§ 21 More metric topology

• Continuous functions and algebraic structures

 (X, τ) : topological sp.

$\Rightarrow$ we want to make sure that algebraic manipulation of functions preserve continuity

$f(x), g(x)$ real-valued cont. funcs. on X
i.e. cont. maps $X \rightarrow \mathbb{R}$

- $f(x) + g(x)$
- $f(x) \cdot g(x)$
- $\frac{f(x)}{g(x)}$ (defined when $g(x) \neq 0$)

Consider $h: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, (s, t) \mapsto s + t$

- $F: X \rightarrow \mathbb{R} \times \mathbb{R}, x \mapsto (f(x), g(x))$ is cont. (Thm 19.6)
- h is cont.
- $h \circ F(x) = f(x) + g(x)$

$\Rightarrow x \mapsto f(x) + g(x)$ is a compos. of cont. maps

$f(x) \cdot g(x)$: same way with $h(s, t) = st$

$f(x)/g(x)$: regard g as $X \rightarrow \mathbb{R} \setminus \{0\}$

use $h: \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}, (s, t) \mapsto s/t$

Rem. same arg. works for $\mathbb{C}, \mathbb{Q}_p = \{a_n p^n + a_{n+1} p^{n+1} + \dots \mid n \in \mathbb{Z}, a_i \in \mathbb{R}\}$

$\mathbb{C}$: usual top. as met. sp.

$\mathbb{Q}_p$: with met. $d(s, t) = p^{-n}$ if $s - t = a_n p^n + a_{n+1} p^{n+1} + \dots$
 $a_n \neq 0$

• Approximation of continuous functions

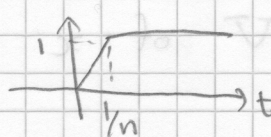
$f_n(x)$ ($n = 1, 2, \dots$) seq. of cont. funcs.

$\Rightarrow$ " $\lim_{n \rightarrow \infty} f_n(x)$ is cont." is (even if lim exists)

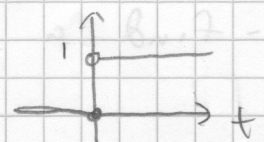
too naive

example

$f_n(t)$



$n \rightarrow \infty$



More abstract setting: $f_n: X \rightarrow Y$

(X, τ) top. sp., (Y, d) met. sp.

Def. $X, Y, f_n, f: X \rightarrow Y$ ($n=1, 2, \dots$)

$(f_n)_{n=1}^\infty$ converges uniformly to f if

$$\forall \varepsilon \exists N \forall n > N, x \in X: d(f_n(x), f(x)) < \varepsilon$$

Rem. we can write $d(f, g) = \sup_{x \in X} d(f(x), g(x))$

for $f, g: X \rightarrow Y$ (possibly $d(f, g) = \infty$)

$$\leadsto d(f, g) + d(g, h) \geq d(f, h), \text{ etc.}$$

unif. conv. means $\forall \varepsilon \exists N \forall n > N, d(f_n, f) < \varepsilon$

$$\text{i.e. } d(f_n, f) \rightarrow 0 \text{ (} n \rightarrow \infty \text{)}$$

Thm 21.6 $(X, \tau), (Y, d), f_n, f: X \rightarrow Y$ as above.

Suppose 1. each f_n is cont.

2. f_n conv. unif. to f

Then f is continuous.

Proof "Same proof as calculus"

estimate $d(f(x), f(y))$

using $d(f_n(x), f_n(y))$, etc.

idea. f_n close to $f \leadsto f(x)$ close to $f_n(x)$, etc.

[Goal If y is close to x , then $f(y)$ is close to $f(x)$]

Formal goal: for any open set U containing $f(x)$
 $f^{-1}(U)$ is an open set (containing x).

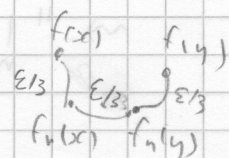
- we may assume $U = B_d(f(x), \varepsilon)$.

$\leadsto$ Need to find a neigh. V of x s.t. $f(V) \subset U$

- find f_n close enough to f .

$$\text{use } N \text{ s.t. } n > N \Rightarrow \forall z, d(f_n(z), f(z)) < \frac{\varepsilon}{3}$$

- find a neigh. V of x



(cont.) after fixing one $n > N$.

continuity of $f_n \Rightarrow \exists$ neigh $V \ni f_n(V) \subset B_d(f_n(x), \varepsilon_3)$

i.e. $y \in V \Rightarrow d(f_n(x), f_n(y)) < \varepsilon_3$

= checking $f(V) \subset B_d(f(x), \varepsilon)$

$d(f(x), f_n(x)) < \varepsilon_3, d(f(y), f_n(y)) < \varepsilon_3$

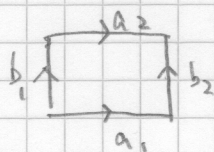
§ 22 Quotient topology

We often define new spaces by "glueing" / "quotient"

Example: torus surface



1. by glueing sides of rectangle



identify a_1 with a_2 , b_1 with b_2

2. starting from $\mathbb{R}^2$, then identifying (x, y) and (x', y') iff $x - x', y - y' \in \mathbb{Z}$

we say $\mathbb{R}^2 / \mathbb{Z}^2$

Making sense of glueing / quot. as a set.

• start from a set X e.g. $[0, 1] \times [0, 1]$, $\mathbb{R}^2$
"union of original pieces"

• define an equivalence relation on X

$x \sim x'$ means x and x' represent the same point in the result.

e.g. $(s, 0) \sim (s, 1), (0, t) \sim (1, t)$

$(s, t) \sim (s+m, t+n)$ for $(m, n) \in \mathbb{Z}^2$

we consider the set of equivalence classes

$$Y = \{[x] : x \in X, [x] = [x'] \text{ iff } x \sim x'\}$$

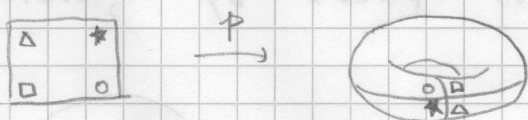
Making sense of quot. as a top. sp.

Suppose: (X, τ) top. sp., $\sim$ equiv. rel. "on X "
 ("gluing rule")

How do we make sense of top. on Y ?

We want the natural map $p: X \rightarrow Y$

$x \mapsto [x]$ to be continuous.



And: can we characterize cont. maps $X \rightarrow Z$
 that appear this way? (up to homeo $Y \simeq Z$)