

§ 22 Quotient topology (continued)

Recap: start from

- top. sp. (X, τ)
- equiv. rel $\sim$ on X ("gluing rule")

 $\leadsto$ consider the quotient set $Y = \{[x] : x \in X\}$

$$[x] = [x'] \Leftrightarrow x \sim x'$$

want: a topology on Y requirement: $X \rightarrow Y, x \mapsto [x]$ cont.

what else do we want?

Def. $(X, \tau_X), (Z, \tau_Z)$ top. sp. $f: X \rightarrow Z$ surjective map

$$\forall z \in Z \exists x \in X : f(x) = z$$

We say f is a quotient map (or stronglycontinuous map) if $\forall U \subset Z$:

$$U \text{ is open } (U \in \tau_Z) \Leftrightarrow f^{-1}(U) \text{ is open } (\in \tau_X)$$

Rem. continuity of f is just " $\Rightarrow$ "" $\Leftarrow$ " let's us say which $U \subset Z$ is openExamples 1. open maps $U \subset X$ open $\Rightarrow f(U) \subset Z$ open2. closed maps $A \subset X$ closed $\Rightarrow f(A) \subset Z$ closed(open or closed) and surjective $\Rightarrow$ quot. mapDef. $U \subset X$ is saturated with resp. to the equiv. rel $\sim$ if: $\forall x \in U, \forall x' \sim x : x' \in U$.Saturated with resp. to $f: X \rightarrow Z$ if:

$$\forall x \in U, \forall x' : f(x') = f(x) \Rightarrow x' \in U$$

Ex. $X = \mathbb{R}^2, x \sim x'$ means $|x| = |x'|$ 

saturated,



not

Prop - Def. (X, τ_X) , $\sim$, Y as in the beginning

$p: X \rightarrow Y$, $x \mapsto [x]$. Then $\exists!$ top. τ_Y on Y

s.t. p is a quot. map; say (Y, τ_Y) is a quot. sp.

Proof. We set $\tau_Y = \{U \subset Y : p^{-1}(U) \text{ open}\}$

Need to check:

1. $S \subset \tau_Y \Rightarrow V = \bigcup_{A \in S} A$ is in τ_Y

2. $S \subset \tau_Y$ finite $\Rightarrow V = \bigcap_{A \in S} A$ is in τ_Y .

1. : take S . $p^{-1}(V) = p^{-1}(\bigcup_{A \in S} A) = \bigcup_{A \in S} p^{-1}(A)$

(we are thinking of: $\{x \in X : \exists A \in S : p(x) \in A\}$)

each $p^{-1}(A) \in \tau_X \Rightarrow$ same for $\bigcup_{A \in S}$

2. : same argument with finite intersection

• if τ'_Y is a top. on Y s.t. $p: X \rightarrow Y$ is

a quot. map. $\tau'_Y = \tau_Y$ □

Another viewpoint: open sets of $Y \Leftrightarrow$ sat. open sets of X

Rem. we have the same claim for surjective maps

$p: X \rightarrow Z$; $x \sim x' \Leftrightarrow p(x) = p(x')$

then $Y \cong Z$ $[x] \mapsto p(x)$
bij.

Examples $Y = \text{torus surf.}$

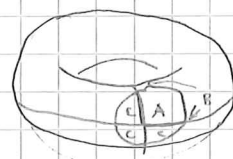
1. $X = [0, 1] \times [0, 1]$, $(0, t) \sim (1, t)$, $(s, 0) \sim (s, 1)$



$A \cup B$ saturated, not open

A open, not saturated.

$A \cup B \cup C$: open & saturated



2. $X = \mathbb{R}^2$,

$(x_1, x_2) \sim (x'_1, x'_2) \Leftrightarrow (x_1 - x'_1, x_2 - x'_2) \in \mathbb{Z}^2$

$\odot \quad \odot^B \quad \odot^B$

A : open, not sat.

p is open (map)

$\odot \quad \odot^A \quad \odot^B$

$A \cup B$: open & sat.

$\odot \quad \odot^B \quad \odot^B$

• Quot vs. subsp.

$A \subset X$, $\sim$: equiv. rel.

$p: X \rightarrow Y$

convexp surj-map

$\rightarrow$ restr. of $\sim$ to A is an equiv. rel.

$q: A \rightarrow p(A)$ is surjective...

But: if T_X is a top. on X , and T_Y is the quot. top. on Y , then $A \xrightarrow{q} p(A)$

might not be a quot. map for the induced top.

Thm. (22.1) Suppose A is saturated
 $A \subset X$, $\sim$, Y , p, q as above. q is a quotient map if any of the following holds

1. A is open in X 1'. A is closed

2. p is an open map 2'. p is closed.

Proof continuity of q is always OK

So we need: $B \subset p(A)$, $q^{-1}(B)$ open in A

$\Rightarrow B$ open in $p(A)$. i.e. $B = U \cap p(A)$

for some open $U \subset Y$

1. $q^{-1}(B) = U \cap A$ for some open $U \subset X$

A open $\Rightarrow U \cap A$ also open in X .

$p^{-1}(B) = q^{-1}(B)$ open $\Rightarrow B$ open in Y

2. again write $q^{-1}(B) = U \cap A$

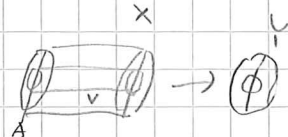
We have $B = p(U \cap A)$

$= p(U) \cap p(A)$

$\uparrow$ open in Y .
 $\{p(x) : x \in U, x \in A\}$ vs.

$\{p(x) : x \in A, \exists x' \sim x : x' \in U\}$

A sat. $\Rightarrow x' \in A$.



Thm 22.2 (part.) $p: X \rightarrow Y$ quot. map. of top.

sp. There is a bij. corresp.

$$\{\text{cont. maps } Y \xrightarrow{f} Z\} \xrightarrow{1:1} \{\text{cont. maps } X \xrightarrow{g} Z\}$$

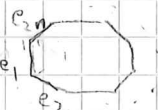
s.t. $p(x) = p(x') \Rightarrow g(x) = g(x')$

for any top. sp. Z ; $g = f \circ p$.

Examples of glueing:

input data: $n = 1, 2, \dots$, partition of $\{1, \dots, 2n\}$
into pairs $\{a_i, b_i\}_{i=1}^n$, $\bigcup_{i=1}^n \{a_i, b_i\} = \{1, \dots, 2n\}$

$X \subset \mathbb{R}^2$: convex $2n$ -gon



$e_1, \dots, e_{2n}$: segments on the boundary

glue e_{a_i} and $e_{b_i} \leadsto$ surface Y .

to be precise: e_i : from vert v_i to v_{i+1}

$$\underbrace{t v_{a_i} + (1-t) v_{a_i+1}}_{e_{a_i}} \sim \underbrace{(1-t) v_{b_i} + t v_{b_i+1}}_{e_{b_i}} \quad 0 \leq t \leq 1$$

curve from $\frac{1}{2}(v_{a_i} + v_{a_i+1})$ to $\frac{1}{2}(v_{b_i} + v_{b_i+1})$

$\leadsto$ loop in Y .

