

§ 23 Connected spaces

or curve

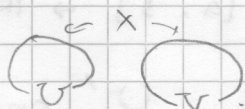
Basic idea: we cannot separate an interval into two pieces without breaking "continuity"



Def. (X, τ) top. sp.

• A separation of X is pair of open sets U, V

s.t. $X = U \cup V$, $U \cap V = \emptyset$, $U \neq \emptyset \neq V$.



• X is connected if there is no separation

Examples 1. $X = [0, 1]$ is connected

(formal proof later.)

2. $X = [0, 1) \cup (1, 2]$ not connected.

3. $X = \mathbb{Q}$ with the induced top from $\mathbb{R}$

is not conn. $U = \{a \in \mathbb{Q} : a < \sqrt{2}\}$

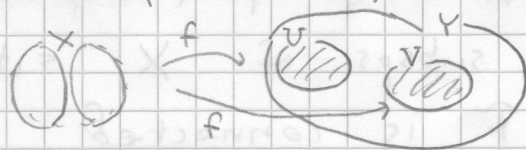
$V = \{a \in \mathbb{Q} : a > \sqrt{2}\}$

Thm 23.5 $(X, \tau_X), (Y, \tau_Y)$ top. sp.

$f: X \rightarrow Y$ cont. map.

If X is connected, then the image $f(X)$ is connected for the induced top.

Proof. Contrapos. $f(X)$ not conn. $\Rightarrow X$ not conn



Take a separation U, V for $f(X)$

i.e. $U = U' \cap f(X)$ for $U' \subset Y$ open, $V = V' \cap f(X)$,

$U \cap V = \emptyset$, $U \neq \emptyset \neq V$, $U \cup V = f(X)$

$\Rightarrow f^{-1}(U) = f^{-1}(U')$, $f^{-1}(V) = f^{-1}(V')$ form a sep. of X

Application : intermediate value theorem.

$f(x)$ cont. func. in real var. $a \leq x \leq b$

r : real number between $f(a)$, $f(b)$

$\Rightarrow \exists c$ between a, b s.t. $f(c) = r$

Proof : again contrapos. Suppose no such c

put $U' = (-\infty, c)$, $V' = (c, \infty)$ open sets of $\mathbb{R}$

• f reps a cont. map $[a, b] \rightarrow \mathbb{R}$

• $f([a, b])$ is covered by U' and V'

$\Rightarrow f([a, b])$ not conn. but $[a, b]$ is conn.

Thm 23.6 $(X_1, T_1), \dots, (X_k, T_k)$: conn. top. sps.

Then $X_1 \times \dots \times X_k$ is connected (for the prod. top.)

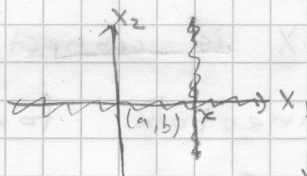
Proof. We can prove this by induction so enough to establish the case of $k=2$.

Fix $a \in X_1$, $b \in X_2$ and put

$$T_x = \{(x', b), (x, y) : x' \in X_1, y \in X_2\}$$

for each $x \in X_1$ i.e. $T_x = (X_1 \times \{b\}) \cup (\{x\} \times X_2)$

$$X_1 \times X_2 = \bigcup_{x \in X_1} T_x$$



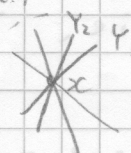
We'll prove : X : top. sp. $(= X_1 \times X_2)$

$\mathcal{F}$: coll. of subsp. of X s.t.

(Thm 23.3) • each $Y \in \mathcal{F}$ is connected

• $\exists x \in X \forall Y \in \mathcal{F} : x \in Y$ ($\bigcap_{Y \in \mathcal{F}} Y \neq \emptyset$)

then $Z = \bigcup_{Y \in \mathcal{F}} Y$ is connected



(cont) then $\circ T_x$ is conn. $F = \{X, x\{b\}, \{x\} \times X_2\}$
 $(x, b) \in$
 $\circ X$ is conn. ; $F = \{T_x : x \in X, \}$
 $\cup (a, b)$

Proof of the claim Contrapos. Z not conn.

$\Rightarrow \exists Y \in F$ not conn.

Take a sep. U, V for Z

$(U = U' \cap Z, V = V' \cap Z$ for open $U', V' \subset X$)
 $U' \cap V' \cap Z = \emptyset, U' \cup V' \supset Z, V' \cap Z \neq \emptyset, \dots)$

We have $x \in U$ or $x \in V$. We may ass. $x \in U$

Take, $Y \in F$ s.t. $Y \cap V \neq \emptyset$

$\circ x \in Y \cap U \Rightarrow Y \cap U \neq \emptyset$

So $Y \cap U, Y \cap V$ give a sep. for Y . $\square$

Thm (cf. 24.1) $\mathbb{R}, [a, b] \subset \mathbb{R}$ $\leftarrow (a < b)$ are connected

we will use the following properties ; X these

1. $A \subset X \Rightarrow \exists t \in X$ s.t. $\begin{cases} \forall a \in A & a \leq t \\ \text{least upper bound} & \text{if } s \in X, \forall a \in A & a \leq s \text{ then } t \leq s \\ \text{supremum.} & \end{cases}$

(2. $\forall x, y : x < y \Rightarrow \exists z : x < z < y$.)

Proof for $X = [a, b]$:

Suppose U, V is a separation of X .
 $\begin{matrix} U & V \\ \cup & \cup \\ a' & b' \end{matrix}$

put $c = \sup U$ then $c \in U$ or $c \in V$

Case 1 $c \in V$; take $d < c$ s.t. $(d, c) \subset V$ $\leftarrow$ open
 then d is an upper bound for U $\rightarrow$ contr.

Case 2 $c \in U$; take $d > c$ s.t. $(c, d) \subset U$ $\leftarrow$ open
 $\begin{matrix} U & V \\ \cup & \cup \\ a' & b' \end{matrix}$

then any $d' \in U$ s.t. $c < d' < d$ satisfies $d' \in U$
 $\rightarrow$ contr.

$X = \mathbb{R}$: sup. U, V is a sep. for $\mathbb{R}$
 $a \in U, b \in V \Rightarrow [a, b]$ would be disconn. $\square$