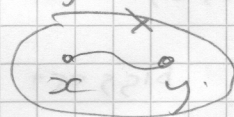


## §24 (path connectedness)

Another way to think about "connectedness" for a top. sp.  $(X, \tau)$ :  $\forall x, y \in X \exists$  "curve" from  $x$  to  $y$

we can "connect"  $x$  to  $y$ .



How to model a curve: continuous map

$f: [0, 1] \rightarrow X$  ("parametrized" curve.)

endpoints:  $f(0), f(1)$

Def.  $(X, \tau)$  is path-connected if  $\forall x, y \in X$

$\exists$  cont. map  $f: [0, 1] \rightarrow X$  s.t.  $f(0) = x, f(1) = y$ .  
(path from  $x$  to  $y$ )

Prop. path-conn.  $\Rightarrow$  conn.

Proof contrapos. not conn.  $\Rightarrow$  not path-conn

Suppose  $U, V$  give a separation of  $X$

$(U \cup V = X, U \cap V = \emptyset, U \neq \emptyset \neq V)$

take  $x \in U, y \in V$  if  $f: [0, 1] \rightarrow X$  is

a cont. map s.t.  $f(0) = x, f(1) = y$ , then

$f^{-1}(U), f^{-1}(V)$  would be a sep. for  $[0, 1]$ . conn.

topologist's sine curve

Rem.  $X = \{(0, y), (x, \sin \frac{1}{x}) : |y| \leq 1, 0 < x \leq 1\}$

is conn. (from Thm 23.4) but not path-conn.

(we cannot connect  $(0, 0)$  to  $(1, \sin 1)$  by a curve in  $X$ )

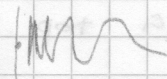
## §25 Components, local connectedness

Q1. When can we say path-conn.  $\Rightarrow$  conn.?

Q2. How do we decompose a top. sp.  $X$  into simpler parts?

A 1. when there is no "local" obstruction

topologist's sine curve: arbitrarily small neighbourhood of  $(0,0)$  is not path-conn.



A 2. take biggest connected / path-conn. subsets

Prop-Def equiv. rel. on top. sp.  $X$ :

1.  $x \sim_c y$ :  $\exists$  connected subsp  $U \subset X$  s.t.  $x, y \in U$ .

2.  $x \sim_{pc} y$ :  $\exists$  path between  $x$  and  $y$ .

checking that these are equiv. rels:

1.  $x \sim_c y, y \sim_c z \Rightarrow x \sim_c z$

i.e.  $x, y \in U, y, z \in V$  for conn.  $U, V \subset X$

$\leadsto$  want  $W \subset X$  conn.,  $x, z \in W$

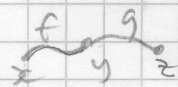
Set  $W = U \cup V$ . ( $y \in U \cap V \Rightarrow W$  conn.)

Thm 83.3

2.  $x \sim_{pc} y, y \sim_{pc} z \Rightarrow x \sim_{pc} z$

i.e.  $f, g: [0,1] \rightarrow X$   $f(0)=x, f(1)=g(0)=y$   
 $g(1)=z$

$\leadsto$  want  $h: [0,1] \rightarrow X$   $h(0)=x, h(1)=z$



treat " $f \cup g$ " as  $h$  i.e.

$$h(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

formally: Thm 18.3 (cf. Sept. 15)

Def. A (connected) component of  $X$ :

an equiv. class of the rel.  $\sim_c$

A path component of  $X$ : an equiv. class of the rel.  $\sim_{pc}$ .

More ex.  $[0,1] \cup [2,3]$   
 $\mathbb{Q} \subset \mathbb{R}$

Ex. The topologist's sine curve has one comp.  
and two path comps.

Rem. A comp. is connected (cf Thm 23.1)

A path comp. is path-conn. :  $y \sim_{pc} x \sim_{pc} z \Rightarrow y \sim_{pc} z$

Def.  $X$  is locally (path-)connected if  $\forall x \in X$   
 $\exists U$  (open) neigh. of  $x \exists V$  (open)  $x \in V \subset U$   
 s.t.  $V$  is (path-)conn.

Thm 25.5  $A$  is a path comp. of  $X$

1.  $\exists B$  comp. of  $X$  s.t.  $A \subset B$   $X$  is  $A \cup B$

2.  $X$  loc. path conn.  $\Rightarrow A$  is a comp. of  $X$

Proof 1.  $x \sim_{pc} y \Rightarrow x \sim_c y$  : take a path  
 $f: [0, 1] \rightarrow X$  between  $x, y \Rightarrow U = f([0, 1])$   
 is a conn. subsp. of  $X \Rightarrow x, y$

2. St. Claim :  $A$  is open

Enough to show :  $\forall y \in A \exists$  open  $V : y \in V \subset A$

(choose such  $V_y$  for each  $y \in A$ )

then  $A = \bigcup_{y \in A} V_y$  (union of open sets)

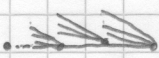
$y$  has a path conn. neigh  $V$  (take  $U = X$   
 in the above def.)  $z \in V \Rightarrow z \sim_{pc} y \Rightarrow z \in A$

St. 2.  $A = B$  for  $B$  in 1.

otherwise  $B$  is a union of more than one  
 path comps. (each open, mutually disj.)

$\Rightarrow$  contradicts connectedness of  $B$ .

Example :  $\mathbb{R}^n$  is loc. path-conn.

"infinite broom"  is path-conn but not  
 locally (path) connected