

§ 26 Compact spaces.

Motivation: closed bounded sets in $\mathbb{R}$, $\mathbb{R}^n$,

examples $[a, b] \subset \mathbb{R}$, $\bar{D} = \{(x, y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$,

1. X does not "spread" to infinity
2. any seq. in X has a convergent subseq.
($\Leftrightarrow$ any infinite subset has a cluster point)
(Sept. 8)

(X, τ) top. sp.

Def. a cover of X : a collection of subsets A

s.t. $X = \bigcup_{Y \in A} Y$ i.e. $\forall x \in X \exists Y \in A : x \in Y$.

an open cover of X : a cover s.t. $\forall Y \in A$
is open.

X is compact for τ if: $\forall$ open cover A
 $\exists$ finite subcoll. $B \subset A$ s.t. $X = \bigcup_{Y \in B} Y$.

Examples, non-examples

1. $\mathbb{R}$ is not compact 

2. $X = \{0, \frac{1}{n} : n = 1, 2, \dots\}$ is compact.

Supp. A is an open cov. of X ,

take $U_0 \in A$ s.t. $0 \in U_0$.

then $\exists N$ s.t. $n > N \Rightarrow \frac{1}{n} \in U_0$.

take $U_1, \dots, U_N$ s.t. $\frac{1}{k} \in U_k$, $B = \{U_0, \dots, U_N\}$

3. $\mathbb{R}^n$ with the Zariski top. (Aug 25, Sept 8) is cpt.

open sets: $U_S = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \forall f \in S, f(x) \neq 0\}$
for subsets S of polynoms.

- A : coll. of open sets $\Rightarrow \bigcup_{U_S \in A} U_S = U_T$, $T = \bigcup_{U_S \in A} S$.

- $\langle S \rangle$ ideal of $\mathbb{R}[t_1, \dots, t_n]$ generated by S .

$\Rightarrow U_{\langle S \rangle} = U_S$

- $\mathbb{R}[t_1, \dots, t_n]$ is Noetherian: any ideal I is finitely generated. $I = (f_1, \dots, f_k)$.

$\leadsto$ given an open cov. $\mathcal{A}$ for the Zariski top.

$I = \left\langle \bigcup_{U \in \mathcal{A}} S \right\rangle$ is fin. gen. $(= (f_1, \dots, f_k))$

each f_i is of the form $\sum_{j=1}^N f_j' f_j''$ $f_j'' \in S$
for some $U_S \in \mathcal{A}$.

$\leadsto$ collect all such S as B .

Thm (26.2). X cpt. sp., $Y \subset X$ closed $\Rightarrow Y$ cpt.

Proof. Take an open cov. $\mathcal{A}$ of Y .

Step 1
set an open cov. of X $\left\{ \begin{array}{l} \text{each } U \in \mathcal{A} \text{ is of the form } U = U' \cap Y \\ U' \subset X \text{ open.} \\ \text{Fix such } U', \text{ set } \mathcal{A}'_0 = \{U' : U \in \mathcal{A}\}. \\ \mathcal{A}' = \mathcal{A}'_0 \cup \{X \setminus Y\} \\ \text{open in } X. \\ \text{then } \mathcal{A}' \text{ is an open cover of } X. \end{array} \right.$

Step 2
use cptness of X $\left\{ \begin{array}{l} \leadsto \text{fin subcoll } \mathcal{B}' \subset \mathcal{A}' \text{ that still covers } X \\ \leadsto \mathcal{B} = \{U : U' \in \mathcal{B}'\} \text{ covers } Y. \quad \square \end{array} \right.$

Thm (26.5) X cpt. sp., $f: X \rightarrow Y$ cont. map

$\Rightarrow f(X)$ is cpt. for the induced top.

Proof. $\mathcal{A}$: open cov. for $f(X)$

$\mathcal{A}' = \{f^{-1}(U) : U \in \mathcal{A}\}$ open cov. for X

$\Rightarrow \exists$ fin. subcoll $\mathcal{B}'$ covering X

$\Rightarrow \mathcal{B} = \{U \in \mathcal{A} : f^{-1}(U) \in \mathcal{B}'\}$ covers $f(X)$.

(to be precise, choose an U_Y s.t. $f^{-1}(U_Y) = Y$)
for each $Y \in \mathcal{B}$ $\square$

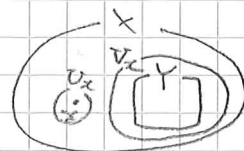
Compactness and Hausdorffness (§17, Sept 8)

Thm 26.3 X Hausdorff sp, $Y \subset X$ compact subsp.
 $\Rightarrow Y$ is closed in X

Proof. Goal: $X \setminus Y$ is open.

Key (Lem 26.4) $\forall x \in X \setminus Y \exists$ open sets U_x, V_x s.t.
 $x \in U_x, Y \subset V_x, U_x \cap V_x = \emptyset$.

Lem $\Rightarrow$ Thm. $X \setminus Y = \bigcup_{x \in X \setminus Y} U_x$.
 union of open sets.



Proof of Lem: for each $y \in Y \exists$ open sets

U'_y, V'_y s.t. $x \in U'_y, y \in V'_y, U'_y \cap V'_y = \emptyset$

(conseq. of Hausdorff prop.).

$\Rightarrow \{V'_y : y \in Y\}$ covers Y ($\{U'_y \cap Y : y \in Y\}$ open cov.)

$\Rightarrow \exists$ finite subcoll $B' = \{V'_{y_1}, \dots, V'_{y_n}\}$ covering Y .

$U_x = \bigcap_{i=1}^n U'_{y_i}, V_x = \bigcup_{i=1}^n V'_{y_i}$ will do.

Thm 26.6 X : cpt., Y : Hausdorff, $f: X \rightarrow Y$
 continuous bijection. $\Rightarrow f$ is homeo.
 (f^{-1} is continuous)

Proof. $g = f^{-1}$ (the inverse map): $Y \rightarrow X$

Enough to check: $A \subset X$ closed $\Rightarrow g^{-1}(A) \subset Y$
 closed.

then $U = X \setminus A$;
 open in X ; $g^{-1}(U) = Y \setminus g^{-1}(A)$ open.

• A is compact (Thm 26.2)

• $f(A)$ is compact (Thm 26.5) hence closed (26.3)

• $f(A) = g^{-1}(A)$

□

Application X : cpt Hausdorff.

$C_{\mathbb{R}}(X) = \{f: X \rightarrow \mathbb{R} \text{ cont. map}\}$, $ev_x(f) = f(x)$

$X' = \{\varphi: C_{\mathbb{R}}(X) \rightarrow \mathbb{R} \text{ alg. hom}\}$ pointwise conv.
 top

fact $X' = \{ev_x : x \in X\}$ as a set.

then we know $X' \cong X$ (homeo) from

- X' Hausdorff.
- $X \rightarrow X'$, $x \mapsto ev_x$ is cont.