

• $[a, b] \subset \mathbb{R}$ is compact (§ 27)

Thm 27.1 Suppose $X \subset \mathbb{R}$ has the property

$$\forall A \subset X \quad \exists x \in X : x = \sup A \quad (\text{least upper b.d. in } X)$$

($\Rightarrow$ X is contained in $[a, b]$ as a subset in X .)

Then X is compact.

Example. $[a, b]$, closed sets of $[a, b]$.

Proof. $\mathcal{A}$: open cover for X i.e.

coll. of open sets for the rel. top. sit. $X = \bigcup_{V \in \mathcal{A}} V$

Want : finite subcoll. $\mathcal{B} \subset \mathcal{A}$ covering X

for $X = [a, b]$ Idea: keep finding $a < c \leq b$ s.t. $[a, c]$ is covered by some fin. subcoll.

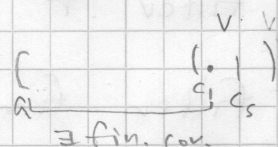
then show $c \rightarrow b$, and $c = b$ is possible.

Step 1. $G = \{c \in X : [a, c] \cap X \text{ as above}\}$

$\Rightarrow \sup G$ exists as $c \in X$.

Step 2 $c_s \in G$

take $V \in \mathcal{A}$ s.t. $c_s \in V$



then $(c_s - \epsilon, c_s + \epsilon) \cap X \subset V$ for some $\epsilon > 0$

$c_s - \epsilon$ is not an upper b.d. of G (in $\mathbb{R}$)

$\Rightarrow \exists c_1 \in G \cap (c_s - \epsilon, c_s)$ (will be c_s)

$\Rightarrow \exists$ fin subcoll. $\mathcal{B}_1$ covering $[a, c_1] \cap X$

$\Rightarrow \mathcal{B}_1 \cup \{V\}$ covers $[a, c_s] \cap X$.

Step 3 $c_s = b$

otherwise we can "go a bit to the right" of c_s .

by $[c_s, c_s + \epsilon) \cap X \subset V$ for $c_s \in V \in \mathcal{A}$.

(if $(c_s, c_s + \epsilon) \cap X = \emptyset$, take $c_2 \in X$ s.t. $[c_s, c_2] \subset V$.)

• Limit for general topological spaces

Idea: specify a collection of "ranges of approximation" around a point, or a subset,...

Def. $x \in X \leadsto$ the neighborhood filter of x is $\mathcal{N}_x = \{ A \subset X : \exists \text{ open } U \text{ s.t. } x \in U \subset A \}$
 ["neighborhood of x "]

similarly: neigh. filter of $B \subset X$:

$$\mathcal{N}_B = \{ A \subset X : \exists \text{ open } U \text{ s.t. } B \subset U \subset A \}$$

$$\text{so } \mathcal{N}_{\bar{x}} = \mathcal{N}_{\{x\}}$$

Properties.

$$1. A \in \mathcal{N}_x, A \subset A' \subset X \Rightarrow A' \in \mathcal{N}_x$$

$$2. \mathcal{B} \subset \mathcal{N}_x \text{ fin. subcoll.} \Rightarrow \bigcap_{A \in \mathcal{B}} A \in \mathcal{N}_x$$

$$3. \emptyset \notin \mathcal{N}_x.$$

Def. (cf. Lem 31.2) a filter on X is a coll. of subsets $\mathcal{F}$ s.t. 1-3 above holds for $\mathcal{F}$.
 (instead of $\mathcal{N}_x$)

a filter $\mathcal{F}$ converges to $x \in X$ if $\mathcal{N}_x \subset \mathcal{F}$

Ex. filter from a sequence: $(x_n)_{n=1}^{\infty}$ seq. in X

$$\leadsto \mathcal{F} = \{ A \subset X : \exists N \text{ s.t. } n > N \Rightarrow x_n \in A \}$$

(A contains almost all x_n)

Def. $x \in X$ is a cluster point of a filter $\mathcal{F}$ if $\forall A \in \mathcal{F} : x \in \bar{A}$.

We can say x cluster pt of $\mathcal{F} \Rightarrow$ "subseq" conv. to x .

formally: $\exists \mathcal{F}'$ filter $\mathcal{F} \subset \mathcal{F}'$, $\mathcal{F}'$ conv. to x .

Def. (32.1) $\mathcal{D}$: coll. of subsets has a finite intersection property if $\mathcal{E} \subset \mathcal{D}$ fin subcoll. $\Rightarrow \bigcap_{A \in \mathcal{E}} A \neq \emptyset$.

Def. an ultrafilter on X is a filter $\mathcal{F}$ s.t.

$$\forall A \subset X : A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F}$$

Prop (cf. Lem 37.1) $\mathcal{D}$ coll of subsets with fin. intersection
then $\exists$ ultrafilter $\mathcal{F}$ s.t. $\mathcal{D} \subset \mathcal{F}$

Proof. (idea) use Zorn's lemma on the system

($\mathcal{E}$: coll of subsets of X , fin int prop, $\mathcal{D} \subset \mathcal{E}$)

take a maximal elem $\mathcal{F}$ in this system.

$\Rightarrow$ check that $\mathcal{F}$ is an ultrafilter using maximality

Cor. $x \in X$ is a cluster pt of filter $\mathcal{F}$

$$\Leftrightarrow \exists \text{ ultrafilter } \mathcal{F}' \text{ s.t. } \begin{cases} \mathcal{F}' \text{ conv to } x \\ \mathcal{F} \subset \mathcal{F}' \end{cases}$$

Thm 26.9 X top sp; X is cpt iff:

$\forall \mathcal{C}$: coll. of closed sets with fin. inter. prop. :

$$\bigcap_{A \in \mathcal{C}} A \neq \emptyset$$

Proof. $\mathcal{C} \leadsto \mathcal{B} = \{V = X \setminus A : A \in \mathcal{C}\}$ coll. of open sets.

want: $\mathcal{C}$ fin. int prop. $\Rightarrow \bigcap_{A \in \mathcal{C}} A \neq \emptyset$

contrapos : $\bigcup_{V \in \mathcal{B}} V = X \Rightarrow \exists \text{ fin. } \mathcal{B}' \subset \mathcal{B} : \bigcup_{V \in \mathcal{B}'} V = X$

$$\left(\bigcap_{A \in \mathcal{C}} A = \emptyset \right) \quad \left(\exists \text{ fin } \mathcal{C}' \subset \mathcal{C} : \bigcap_{A \in \mathcal{C}'} A = \emptyset \right) \quad \square$$

Prop X is cpt iff : $\forall$ filter $\mathcal{F} \exists x \in X$ cluster pt of $\mathcal{F}$

Proof $\exists$ cluster pt for $\forall$ filter $\Rightarrow$ cpt.

take $\mathcal{C}$ coll. of closed sets w/ fin. int. prop.

want $\bigcap_{A \in \mathcal{C}} A \neq \emptyset$.

Prop. (Lem 37.1) gives a filter $\mathcal{F} \supset \mathcal{C} \leadsto$ cluster pt x .

$$x \in \bar{A} \text{ for all } A \in \mathcal{F}. \text{ For } A \in \mathcal{C} \quad \bar{A} = A$$

cpt $\Rightarrow \exists$ clns. pt. for $\forall$ filter ; contrapos.

take a filter $\mathcal{F}$ without a cluster pt. , $\mathcal{C} = \{\bar{A} : A \in \mathcal{F}\}$
has fin int prop, $\bigcap_{A \in \mathcal{C}} \bar{A} \neq \emptyset$

Cor. X cpt $\Leftrightarrow \forall$ ultrafilter $\mathcal{F} \exists x \in X : \mathcal{F}$ conv. to x .

Tychonoff's theorem (39.3) $(X_\alpha, \tau_\alpha)_{\alpha \in I}$ compact

top. spaces $\Rightarrow \prod_{\alpha \in I} X_\alpha$ is compact for the prod. top.

Proof. check: $\mathcal{F}$ ultrafilter on $\prod_{\alpha} X_\alpha \Rightarrow \exists (x_\alpha)_{\alpha \in I}$
s.t. $\mathcal{F}$ conv. to $(x_\alpha)_\alpha$.

Step 1. finding $x_\alpha : \mathcal{F}_\alpha = \{ A \subset X_\alpha : \pi_\alpha^{-1}(A) \in \mathcal{F} \}$
 $\uparrow$ proj. map.

$\Rightarrow \mathcal{F}_\alpha$ is a filter, hence has a cluster pt x_α .

Step 2. $\mathcal{F}$ conv. to $(x_\alpha)_\alpha$. ($\Leftrightarrow (x_\alpha)_\alpha$ is a cluster pt.)

neighborhood in prod. top: contains

$$\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_k}^{-1}(U_{\alpha_k}) \quad \text{for } U_{\alpha_k} \text{ open } \ni x_{\alpha_k} \\ k = 1, 2, \dots$$

$\pi_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$: otherwise compl. $= \pi_{\alpha_i}^{-1}(\text{compl. of } U_{\alpha_i})$
is in $\mathcal{F}$, x_{α_i} not cluster pt of $\mathcal{F}_\alpha$.

then $\bigcap_{i=1}^k \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$. $\square$