

Zorn's lemma

- $(S, \leq)$ partially ordered set ;
- $x \leq y$ and $y \leq x$ (if defined) $\Rightarrow x = y$
- $x \leq y$ and $y \leq z$ (if defined) $\Rightarrow x \leq z$

Suppose $\forall T \subset S$ s.t. $(T, \leq)$ is totally ordered

$$(t, t' \in T \Rightarrow t \leq t' \text{ or } t' \leq t) \Rightarrow \exists a \in S \quad \forall t \in T : t \leq a$$

(upperbound of T)

Then S has maximal elements $a : x \in S \Rightarrow x \leq a$
or no order rel between x and a .

Typically : $S = (\text{subsets } A \subset X \text{ with some structure})$

or $(\mathcal{T} = \{A \subset X : \dots\} \text{ with some cond.})$

- $x \leq y$ means inclusion $x \subset y$.
- upper bound of $T : \bigcup_{x \in T} x$

Sketch of proof (by contradiction)

Suppose no max elem exist

- for each $T \subset S$ s.t. $(T; \leq)$ tot. ord.
choose $f(T) \in S$ s.t. $\exists$ upper bd a of T
 $f(T) \not\leq a$.

- define x_α for ordinal numbers α .
($\alpha \leftrightarrow$ well-ordered sets A ; $\forall B \subset A \exists \min B \in A$)
 $x_0 \in S$ any elem, $x_\alpha = f(\{x_\beta : \beta < \alpha\})$
- $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$, defined for all ordinals
is too much. (contradicts at $|\alpha| > |S|$)