

§ 28 Limit point compactness.

Def. (X, T) top. sp. It is sequentially compact

if $\forall$ seq. $(x_n)_{n \in \mathbb{N}}$ in $X \exists$ subseq. $(y_i)_{i \in \mathbb{N}}$; $y_i = x_{n_i}$ that converges to some point $y \in X$.

Thm 28.2 (X, d) metric space.

X cpt (as top sp.) $\Leftrightarrow X$ seq. cpt.

Proof $\Rightarrow$: take a seq. $(x_n)_n$. consider the filter

$\mathcal{F} = \{ A \subset X : A \text{ contains } x_n \text{ for almost all } n \}$

and an ultrafilter $\mathcal{F}' \supset \mathcal{F}$.

St.1 $\leadsto$ $\exists y$ s.t. $\mathcal{F}'$ conv. to y . i.e. any neighbourhood of y is in $\mathcal{F}'$.

St.2 $\forall A \in \mathcal{F}'$ $x_n \in A$ for infinitely many n .

(otherwise $X \setminus A \in \mathcal{F} \subset \mathcal{F}'$, set $\emptyset = A \cap (X \setminus A)$ in $\mathcal{F}'$)

$\leadsto \forall \varepsilon > 0$ $B_d(y, \varepsilon)$ contains x_n for some n .

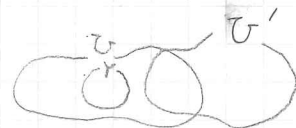
$\Leftarrow$: Take an open cover $\mathcal{A}$ for X

want: finite subcoll. $\mathcal{B}$ that covers X

Step 1. (Lebesgue number lemma, 29.5) $\exists \delta > 0$ s.t.

$\forall Y \subset X$ s.t. $\forall y, y' \in Y$ $d(y, y') < \delta$
(diam. $Y \leq \delta$)

$\exists U \in \mathcal{A}$ s.t. $Y \subset U$.



Proof by contr: $\forall \delta \exists Y \subset X$ as above $\forall U \in \mathcal{A} : Y \not\subset U \Rightarrow$ contr.

for each n , take $C_n \subset X$ s.t. $\begin{cases} \text{diam } C_n \leq \frac{1}{n} \\ \forall U \in \mathcal{A} \quad C_n \not\subset U \end{cases}$

and fix $x_n \in C_n$.

By seq. cpt-ness $\exists a \in X$ subseq. $y_i = x_{n_i}$ s.t. $y_i \rightarrow a$.

take $U \in \mathcal{A}$ s.t. $a \in U$. take $\varepsilon : B_d(a, \varepsilon) \subset U$.

When $\frac{1}{n} < \frac{\varepsilon}{2}$, $d(a, x_n) < \frac{\varepsilon}{2}$, then $C_n \subset B_d(x, \frac{\varepsilon}{2}) \subset B_d(a, \varepsilon)$



hence $c_n \in \bigcup \mathcal{A}$, contr.

Step 2. Cover X by small balls.

Set $\varepsilon = \delta/3$. Claim: $\exists x_1, \dots, x_k$ s.t.

$$X = \bigcup_{i=1}^k B_d(x_i, \varepsilon) \quad ; \text{ proof by contr.}$$

Suppose no such $x_1, \dots, x_k$ exist.

$\Rightarrow$ keep choosing $x_1, x_2, \dots$ s.t. x_{n+1} is not in $\bigcup_{i=1}^n B_d(x_i, \varepsilon)$.

$\Rightarrow$ contradicts with seq. cptness.

Step 3. Find a finite subcoll. $B \subset \mathcal{A}$.

Use $x_1, \dots, x_k$ from Step 2.

$$B_d(x_i, \varepsilon) \text{ has diam} \leq 2\varepsilon = \frac{2\delta}{3}$$

$$\Rightarrow \exists U_i \in \mathcal{A} \text{ s.t. } B_d(x_i, \varepsilon) \subset U_i \quad ; \quad B = \{U_i\}_{i=1}^k \quad \square$$

§29 Locally compact spaces.

Def. (X, τ) is locally compact if $\forall x \in X$

$$\exists U \subset C \subset X \text{ s.t. } U \text{ open } \ni x$$

C cpt. for the induced top.

$$\text{Ex. } \mathbb{R}^n \quad ; \quad (x_1, \dots, x_n) \in \prod_{i=1}^n (x_i - \varepsilon, x_i + \varepsilon) \subset \prod_{i=1}^n [x_i - \varepsilon, x_i + \varepsilon]$$

Prop-Def. (X, τ) loc. cpt. ; $Y = X \cup \{\infty\}$

∞ : an extra point not in X

Y is a compact top. sp. with open sets

$$\tau' = \{ U \subset X : \text{open} \} \cup \{ \{\infty\} \cup (X \setminus K) : K \subset X \text{ cpt} \}$$

(Y, τ') is called the one-point compactification of X

$$\text{Ex. } X = \mathbb{R} \rightsquigarrow Y \cong_{\text{homeo}} S^1$$

$$X = \mathbb{R}^n \rightsquigarrow Y \cong S^n$$



cont.) Proof of claim

Step 1. T' is a topology

$$\text{Set } T'_1 = T, \quad T'_2 = \{\{\infty\} \cup (X \setminus K) : K \in X \text{ cpt}\}$$

$$\text{so } T' = T'_1 \cup T'_2$$

- T'_2 is closed under union, ^(other than empty union) fin. intersection

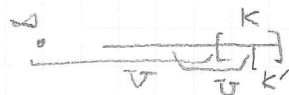
$\Rightarrow \{K : K \in X \text{ cpt}\}$ closed under intersection
fin. union.

intersection: closed sets of cpt sets are cpt.

fin. union: OK for cptness

- union of $U \in T'_1, V \in T'_2$; $V = \{\infty\} \cup (X \setminus K)$

$$U \cup V = \{\infty\} \cup (X \setminus K'), \quad K' = K \setminus U$$



- intersection of $U \in T'_1, V \in T'_2$

$$U \cap V = U \setminus K = U \cap (X \setminus K)$$

Step 2. (Y, T') is cpt.

Take open cover $\mathcal{A}$ for Y . take $V \in \mathcal{A}$

st. $\infty \in V$. then $V \in T'_2$ i.e. $V = \{\infty\} \cup (X \setminus K)$

K cpt. in X

K is covered by $\{U \cap K : U \in \mathcal{A}\}$

$U \cap K$ is open for the ind. top.

- obv. if $U \in T'_1$

- if $U \in T'_2$ so $U = \{\infty\} \cup (X \setminus K')$

$$\text{then } U \cap K = K \cap \underbrace{(X \setminus K')}_{\text{open.}}$$

$\leadsto \exists$ fin subcoll B' st. $\{U \cap K : U \in B'\}$ cov. K .

$\leadsto B = B' \cup \{V\}$ covers Y . $\square$