

§ 43 Complete metric spaces.

Def. (X, d) met. sp. $(d(x, y) = d(y, x) \geq 0, d(x, y) + d(y, z) \geq d(x, z))$

- a Cauchy sequence is a seq. $(x_n)_{n=1}^{\infty}$

in X s.t. $d(x_m, x_n) \rightarrow 0 \quad (m, n \rightarrow \infty)$

i.e. $\forall \varepsilon > 0 \exists N : m, n > N \Rightarrow d(x_m, x_n) < \varepsilon$

- (X, d) is complete if any Cauchy seq. has a limit $\exists y \in X \quad d(x_n, y) \rightarrow 0 \quad (n \rightarrow \infty)$

Ex. $\mathbb{R}^k$, $d(x, y) = \sqrt{\sum_{i=1}^k |x_i - y_i|^2}$ is complete

(follows from cpt-ness of $\prod_{i=1}^k [a_i, b_i]$)

Other examples.

$C_0(\mathbb{R}; \mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R}, f(t) \rightarrow 0 \quad (|t| \rightarrow \infty)\}$
with $\|f\|_{\infty} = \max_t |f(t)|$