

§ 51 Homotopy of paths.

We want to look at curves / paths in a space.

- don't care about precise parametrization (if the "path" is the same)
- allow continuous deformation



Parameter space for each path: $I = [0, 1]$

i.e. a path in X is a cont map $f: I \rightarrow X$

Continuous deformation of paths (f_t)

t : deform. param. $\rightarrow$ again we use I for the range of t .

Def. $x_0, x_1 \in X$, $f, f': I \rightarrow X$ cont. maps, s.t.
 $f(0) = x_0 = f'(0)$, $f(1) = x_1 = f'(1)$

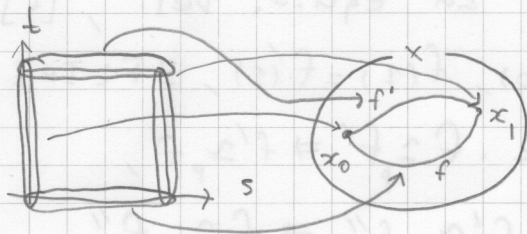
We say f and f' are path-homotopic if

$\exists$ cont. map. $F: I \times I \rightarrow X$ ("cont. family of paths")

s.t. $F(s, 0) = f(s)$, $F(s, 1) = f'(s)$

F represents a $\swarrow$ from f to f'

$F(0, t) = x_0$, $F(1, t) = x_1$; endpts are fixed



F : path homotopy
between f and f'

Related concepts.

- Homotopy between cont. maps $f, f': X \rightarrow Y$

$F: X \times I \rightarrow Y$ s.t. $F(x, 0) = f(x)$, $F(x, 1) = f'(x)$

Cauton: for paths we put more conditions

$f(0) = F(0, t) = f'(0)$, $f(1) = F(1, t) = f'(1)$

- when X is locally cpt, such F corresponds to a map $\tilde{F}: I \rightarrow \mathcal{C}(X, Y)$ (Thm 46.11)
 $\{\{f: X \rightarrow Y \text{ cont.}\}\}$

for the compact-open topology on $\mathcal{C}(X, Y)$

↪ start from the coll. of subsets of $\mathcal{C}(X, Y)$

$$\mathcal{A}_0 = \{S(c, U) = \{f: f(c) \in U\} : c \in X \text{ cpt}, U \subset Y \text{ open}\}$$

then take fin. int. $\mathcal{A}_1 = \{S_1 \cap \dots \cap S_k : S_i \in \mathcal{A}_0\}$

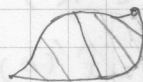
and use this as base of top. on $\mathcal{C}(X, Y)$.

Ex. 1. $X \subset \mathbb{R}^n$ convex subset: $a, b \in X, 0 \leq t \leq 1 \Rightarrow (1-t)a + tb \in X$

↪ any paths $f, f': I \rightarrow X$ s.t. $f(0) = f'(0), f(1) = f'(1)$

are path-homotopic: $F: I \times I \rightarrow X$ given by

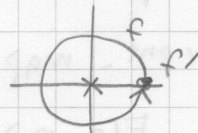
$$F(s, t) = (1-t)f(s) + tf'(s)$$



2. $X = \mathbb{R}^2 \setminus \{(0, 0)\}$, $x_0 = (1, 0) = x_1$

$$f(s) = (\cos 2\pi s, \sin 2\pi s)$$

not path-hom. to $f'(s) = (1, 0)$



Prop-Def. write $f \simeq_p f'$ when paths $f, f': I \rightarrow X$ are path-hom. This is an equiv. rel., $[f]$: equiv. class.
 (i.e. $f \simeq_p f' \Leftrightarrow f(0) = f'(0), f(1) = f'(1), \exists F: I \times I \rightarrow X, \dots$)

Proof want 1. $f \simeq_p f, 2. f \simeq_p f' \Rightarrow f' \simeq_p f,$
 3. $f \simeq_p f', f' \simeq_p f'' \Rightarrow f \simeq_p f''$

1. $F(s, t) = f(s)$ will do.

2. "flip" the deform param t ; take a path-hom F

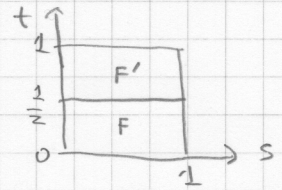
$$F(s, 0) = f(s), F(s, 1) = f'(s), F(0, t) = f(0) = f'(0), F(1, t) = \dots$$

then $F'(s, t) = F(s, 1-t)$ will do.

3. $\left. \begin{matrix} f \\ f' \end{matrix} \right\}$ path-hom bet. $\left\{ \begin{matrix} f \text{ \& } f' \\ f' \text{ \& } f'' \end{matrix} \right.$ $F(s, 0) = f(s), F(s, 1) = f'(s)$
 $F'(s, 0) = f'(s), F'(s, 1) = f''(s)$

$\leadsto$ define $F'' : I \times I \rightarrow X$ by

$$F''(s, t) = \begin{cases} F(s, 2t) & (0 \leq t \leq \frac{1}{2}) \\ F'(s, 2t-1) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$



this is a path-hom. between f and f'' $\square$

Def. f : path in X , $f(0) = x_0, f(1) = x_1$
 g : path in X , $g(0) = x_1, g(1) = x_2$

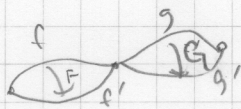
the composition (or concatenation) of f and g :

the path h in X ; $h(s) = \begin{cases} f(2s) & (s \leq \frac{1}{2}) \\ g(2s-1) & (s \geq \frac{1}{2}) \end{cases}$

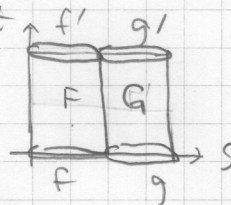
we write $h = f * g$ ($g * f$ is also common)

Prop. when $h = f * g$, $[h]$ only depends on $[f], [g]$

Proof. want $f \simeq_p f', g \simeq_p g' \Rightarrow f * g \simeq_p f' * g'$



by



Suppose $\left. \begin{matrix} F \\ G \end{matrix} \right\}$ path hom between $\left\{ \begin{matrix} f \text{ and } f' \\ g \text{ and } g' \end{matrix} \right.$

$$\text{Put } H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq \frac{1}{2} \\ G(2s-1, t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

$$\text{then } H(s, 0) = f * g, H(s, 1) = f' * g',$$

$$H(0, t) = F(0, t) = f(0), H(1, t) = G(1, t) = g(1)$$

$\leadsto$ we can write $[f] * [g] = [f * g]$

Not. $\pi_1(X) = \{ [f] : f \text{ path in } X \}$

fundamental groupoid of X .