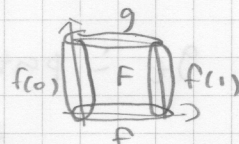


## § 5.2 Fundamental group (oid)

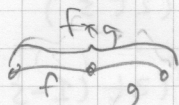
Recap:

- path in  $X$  : cont map  $f: I \rightarrow X$   
 $I = [0, 1]$
- path-homotopy of paths :  $f \simeq_p g$  if  $f(0) = g(0)$ ,  
 $f(1) = g(1)$ ,  $\exists F: I \times I \rightarrow X$

"cont. deform. of paths" Sit.



- composition of paths; when  $f(1) = g(0)$



$$[f] * [g] = [f * g] \quad \text{well-defined}$$

$\uparrow$  path-homotop. class

We want to see that  $\pi_1(X) = \{[f] : f \text{ path in } X\}$  is a groupoid.

Thm 51.2.  $f, g, h$  path in  $X$

1. when  $f(1) = g(0)$ ,  $g(1) = h(0)$

$$([f] * [g]) * [h] = [f] * ([g] * [h]) \quad \text{associativity}$$

2. given  $x \in X$ , consider the const path  $e_x: I \rightarrow X$   
at  $x$  ( $e_x(s) = x$  for  $0 \leq s \leq 1$ )

$$f(0) = x \Rightarrow [e_x] * [f] = [f]$$

$$f(1) = x \Rightarrow [f] * [e_x] = [f]$$

units / identities

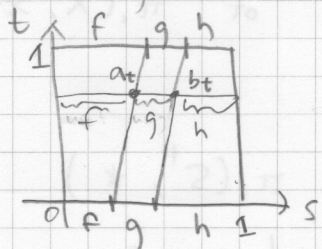
3. put  $\tilde{f}(s) = f(1-s)$  (flipped path)

then  $[f] * [\hat{f}] = [e_x]$  for  $x = f(o)$

$$[\tilde{f}] * [f] = [e_y] \quad \text{for } y = f(1)$$

Proof 1. we need a path homotopy between

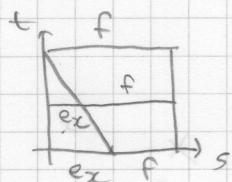
$$(f * g) * h \quad \text{and} \quad f * (g * h)$$



$$F(s, t) = \begin{cases} f(s, \text{rescaled}) & 0 \leq s \leq a_t \\ g(s, \text{rescaled, offset}) & a_t \leq s \leq b_t \\ h(s, \text{rescaled, offset}) & b_t \leq s \leq 1 \end{cases}$$

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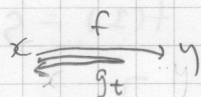
no. 2

2. Path homotop from  $e_x * f$  to  $f$  ( $f(0) = x$ )

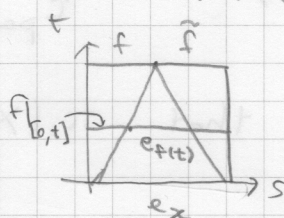
$$F(s, t) = \begin{cases} x & s \leq \frac{1-t}{2} \\ f(s \text{ rescaled, offset}) & s \geq \frac{1-t}{2} \end{cases}$$

3. Path homotop from  $e_x$  to  $f * \tilde{f}$  ( $f(0) = x$ )

$g_t$  : path from  $x$  to  $x$ , go until  $f(t)$  then come back



$$g_t(s) = \begin{cases} f(2ts) & s \leq \frac{1}{2} \\ f(2t(1-s)) & s \geq \frac{1}{2} \end{cases}$$



$$\leadsto F(s, t) = g_t(s)$$

Now :  $\pi_1(X)$  is (the arrow set of) a groupoidwith base set  $X$  :

- source of  $[f] = f(0)$ , target of  $[f] = f(1)$
- product  $[f] * [g]$  (defined when  $t(f) = s(g)$ ), associative)
- unit arrow of  $x \in X$  :  $[e_x]$  ( $[e_x] * [f] = [f]$ ,  $[f] * [e_{f(0)}] = [f]$ )
- inverse  $[f]^{-1} = [\tilde{f}]$  ( $[f] * [f]^{-1} = [e_{s(f)}]$ ,  $[f]^{-1} * [f] = [e_{t(f)}]$ )

Prop-Def.  $x \in X$ .  $\hookrightarrow$  loop based at  $x$ 

$$\pi_1(X, x) = \{ [f] : \text{path in } X, f(0) = x = f(1) \} \subset \pi_1(X)$$

is a group, called the fundamental group of  $X$  relative to the base point  $x$

Proof.  $[e_x]$  is the unit elem of  $\pi_1(X, x)$ .  $\square$ 

Examples (facts)

$$1. \pi_1(\mathbb{R}^2 \setminus \{0, 0\}, (1, 0)) \cong \mathbb{Z} \cong \pi_1(S^1, *)$$

number of times the loop goes around the center

2.  $X = \mathbb{R}^2 / \mathbb{Z}^2$  (torus surface)

$$\pi_1(X, (0,0)) = \mathbb{Z}^2$$

3.  $\pi_1(S^2, *) \cong \{0\}$

4.  $X$  : higher genus surface

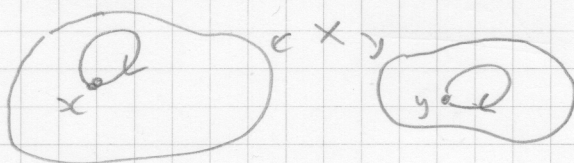


num. holes  $\geq 2$

$\Rightarrow \Gamma = \pi_1(X, *)$  acts on the hyperbolic plane  $\mathbb{H}$  by isometries,  $X \cong \mathbb{H}/\Gamma$ .

Dependence on base point:

$\pi_1(X, x)$  depends on  $x \in X$ , and if  $y \in X$  is "unreachable" from  $x$  by paths, no relation between  $\pi_1(X, x)$  and  $\pi_1(X, y)$



Thm 52.1 Suppose  $\alpha$  is a path from  $x$  to  $y$

Then  $\hat{\alpha} : \pi_1(X, y) \rightarrow \pi_1(X, x)$ ,  $[f] \mapsto [\alpha] * [f] * [\alpha]^{-1}$   
 $y \xrightarrow{f} y \quad x \xrightarrow{\alpha} y \xrightarrow{f} y \xrightarrow{\alpha^{-1}} x$   
 is an isomorphism

Proof : Step 1.  $\hat{\alpha}$  is a homomorphism

$$\text{want } \hat{\alpha}([f]) * \hat{\alpha}([g]) = \hat{\alpha}([f] * [g])$$

$$[\alpha] * [f] * [\alpha]^{-1} * [\alpha] * [g] * [\alpha]^{-1} \quad \text{vs.} \quad [\alpha] * [f] * [g] * [\alpha]^{-1}$$

$\underbrace{[\alpha] * [f] * [\alpha]^{-1} * [\alpha]}_{[e_y]} \text{ by } [\alpha]^{-1} = [\alpha]^{-1}$

Step 2.  $\hat{\alpha}$  has an inverse map.

$\hat{\alpha}^{-1}$  is the inverse

Rem. this proof works for any groupoid

Cor.  $X$  path-connected. ( $\forall x, y \in X \exists$  path from  $x$  to  $y$ )

$\Rightarrow \pi_1(X, x)$  ( $x \in X$ ) are all isomorphic.

Def.  $X$  is simply connected if  $\forall x \in X \pi_1(X, x) \cong \{0\}$   
 (often assume path-conn.; then  $\exists x \in X \dots$ )

## Induced homomorphism (functoriality).

Prop-Def.  $h: X \rightarrow Y$  cont map.  $x \in X, y = h(x)$ .

$$h_*: \pi_1(X, x) \rightarrow \pi_1(Y, y), [f] \mapsto [h \circ f]$$

$I \rightarrow X \qquad \qquad I \rightarrow X \rightarrow Y$

is a homomorphism, called the hom induced by  $h$ Proof.  $(h \circ f) * (h \circ g) = h \circ (f * g)$ . (compat. with prod.)

$$\Rightarrow f \simeq_p f' \Rightarrow h \circ f \simeq_p h \circ f'$$

by  $F \qquad \qquad \qquad$  by  $h \circ F$

Thm 52.4.  $h: X \rightarrow Y, k: Y \rightarrow Z, x \in X, y = h(x), z = k(y)$ .Then  $k_* \circ h_* = (k \circ h)_*$  as homs  $\pi_1(X, x) \rightarrow \pi_1(Z, z)$