

§53 Covering spaces

We've seen open covering (coll. ^{of} open sets $\{U_i\}$ s.t. $\bigcup U_i = X$) for compactness.

Another useful idea: a bigger (but simpler) space "wrapping around" a smaller (but more complicated) one

Ex. $\mathbb{R} \rightarrow S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, t \mapsto (\cos 2\pi t, \sin 2\pi t)$

Def. E, B top. sp., $p: E \rightarrow B$ cont. surj. map
 $U \subset B$ open set; we say U is evenly covered by p if

- the inv. img. $p^{-1}(U)$ is a disj. union $\bigsqcup_{\alpha \in I} V_\alpha$

for open sets $V_\alpha \subset E$ "slice"

- restriction of p to V_α is a homeo. $V_\alpha \rightarrow U$ for each α .

We say $E \xrightarrow{p} B$ is a covering map (covering space)

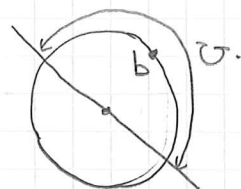
if $\forall b \in B \exists$ open $U \subset B : b \in U, U$ is evenly covered by p .

Thm. 53.1 the above ex. $\mathbb{R} \rightarrow S^1$ is a covering map.

Proof. fix $b = (\cos 2\pi \alpha, \sin 2\pi \alpha)$.

choice of $U : \{(\cos 2\pi(\alpha+t), \sin 2\pi(\alpha+t)) : -\frac{1}{4} < t < \frac{1}{4}\}$

$$p^{-1}(U) = \bigsqcup_{k=-\infty}^{\infty} \underbrace{\left(\alpha + k - \frac{1}{4}, \alpha + k + \frac{1}{4}\right)}_{V_k \subset \mathbb{R} \text{ open.}}$$



restriction of p to V_k is

bijjective $V_k \rightarrow U$, homeomorphism

□

Rem. Some operations for cov. maps

1. $E \xrightarrow{p} B$ cov., $B_0 \subset B \rightsquigarrow E_0 \xrightarrow{p} B_0$ cov. for $E_0 = p^{-1}(B_0)$

2. $E_1 \xrightarrow{p_1} B_1, E_2 \xrightarrow{p_2} B_2$ cov. $\rightsquigarrow E_1 \times E_2 \xrightarrow{p_1 \times p_2} B_1 \times B_2$ cov.

§ 54 Covering and fundamental group.

Goal : see $\pi_1(S^1, *) \cong \mathbb{Z}$ from the covering $\mathbb{R} \xrightarrow{p} S^1$

Idea : - fix basepoint $0 \in \mathbb{R}$ so $p(0) = * \in S^1$.

- given $[f] \in \pi_1(S^1, *)$, find a lift path $\tilde{f}$ in $\mathbb{R}$ that starts from 0 $\nearrow p \circ \tilde{f} = f$
- $\tilde{f}(1)$ is not 0, but another int. ($p(\tilde{f}(1)) = *$)
 $\leadsto$ set corresp. $\pi_1(S^1, *) \rightarrow \mathbb{Z}, [f] \mapsto \tilde{f}(1)$

Def. $E \xrightarrow{p} B$ cov. map, $f: X \rightarrow B$ cont. map.

a lift(ing) of f is a cont map $\tilde{f}: X \rightarrow E$ s.t. $p \circ \tilde{f} = f$

Setting $b_0 \in B, e_0 \in E, p(e_0) = b_0$

Lem 54.1. f : path in $B, f(0) = b_0$ Then $\exists$ unique path $\tilde{f}$ in E s.t. $\begin{cases} \tilde{f} \text{ is a lift of } f \\ \tilde{f}(0) = e_0 \end{cases}$

Lem 54.2 $F: I \times I \rightarrow B, F(0,0) = b_0$ Then $\exists$ unique

lift $\tilde{F}: I \times I \rightarrow E$ s.t. $\tilde{F}(0,0) = e_0$

when F is a path homotopy ($F(0,t)$ const, $F(1,t)$ const)

then this $\tilde{F}$ is also a path homotopy.

Cor (Thm 34.3) f, g path in $B, f(0) = b_0 = g(0)$

$f \simeq_p g \Rightarrow \tilde{f} \simeq_p \tilde{g}$ for the lifts as in Lem 54.1

Def. Lifting correspondence: $\phi: \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$



Th'm 54.4 (1) if E is path-connected ($\forall x, y \in E$

$\exists$ path from x to y) then the above ϕ is surjective

(2) if E is simply connected ($\pi_1(E, x) = 0$) then

moreover ϕ is bijective.

Proof (1), take $x \in p^{-1}(b_0)$. $\tilde{f}$: path from e_0 to x .

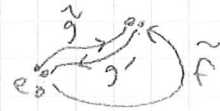
Set $f = p \circ \tilde{f}$. then $\tilde{f}$ is the lift of f s.t. $\tilde{f}(0) = e_0$.

$\Rightarrow \phi([f]) = \tilde{f}(1) = x$.

cont.) Proof (2) take $[f], [g] \in \pi_1(B, b_0)$ s.t.

$d([f]) = d([g])$, take lifts $\tilde{f}, \tilde{g}$, $\tilde{f}(0) = e_0 = \tilde{g}(0)$

g' : inverse path of $\tilde{g}$



then $\tilde{f} * g'$ is a loop starting at e_0 .

E simply connected $\Rightarrow \tilde{f} * g'$ is homotopic to the const. path at e_0

$$\Leftrightarrow \tilde{f} \simeq_p \tilde{g}$$

$$\uparrow [g] = [e_0] * [g] = [\tilde{f}] * [g'] * [\tilde{g}] = [\tilde{f}].$$

Then $p_*([f]) = p_*([g])$ i.e. $[f] = [g]$

Cor. $\pi_1(S^1, *) \simeq \mathbb{Z}$.

