

Def. (Exercise 51.3) X is contractible if
 the identity map $\text{id}_X : X \rightarrow X$ is homotopic to a const
 map $f : X \rightarrow X$, $f(x) = x_0$ for some $x_0 \in X$.
 (any) id_X is null homotopic



Ex. $\mathbb{R}^k$, convex sets of $\mathbb{R}^k$, ...

Contractible $\Rightarrow$ path-connected, simply connected

If B has a covering map from a contractible space
 E , we say that B is "the" classifying space
 of $\Gamma = \pi_1(B, *)$ (B will be path-conn.)

I write $B = B\Gamma$

Ex. $S^1 = B\mathbb{Z}$, $(S^1 \times \dots \times S^1)^{k \times} = B\mathbb{Z}^k$, $\Sigma g = B\Gamma_g$
 but S^k ($k > 1$) is not $B\Gamma$

Proof of Lems 54.1, 54.2 ; more general version.

Y : top sp. $F : Y \times I \rightarrow B$, $\tilde{f}_0 : Y \rightarrow E$
 Given s.t. $p \circ \tilde{f}_0(y) = F(y, 0)$, $\exists! \tilde{F} : Y \times I \rightarrow E$, $p \circ \tilde{F} = F$, $\tilde{F}(y, 0) = \tilde{f}_0(y)$

(54.1 : $Y = \{*\}$, $\tilde{f}_0 : Y \rightarrow E$, $\tilde{f}_0(*) = x_0$
 54.2 : $Y = I$, $\tilde{f}_0 = \tilde{f}$ given by 54.1

Step I. local const. of $\tilde{F}$ around $\{y\} \times I$ $y \in Y$.

U_t : evenly covered neigh. of $F(y, t)$.

$N_{y,t}$: neigh of y , N'_t : neigh of t s.t.

$$N_{y,t} \times N'_t \subset F^{-1}(U_t)$$

By compactness $\exists t_1, \dots, t_k$ s.t. $I = \bigcup_{i=1}^k N_{t_i}$

$$\text{Set } N_y = \bigcap_{i=1}^k N_{y,t_i} \quad t_1 < t_2 < \dots < t_k$$

$\Rightarrow \tilde{F} : N_y \times I \rightarrow E$ is defined by choosing the "correct
 slice" for $\tilde{F}(y, t_i)$ from $i=1$ to $i=k$.
 determined by $\tilde{f}_0(y)$

Step 3 $\tilde{F}$ well defined for all $Y \times I$.

$\tilde{F}|_{N_y \times I}$ defined for each y .

Check that they agree on $(N_y \cup N_{y'}) \times I$ for y, y'

$\Rightarrow$ follows from uniqueness

Step 2 $\tilde{F}|_{N_y \times I}$ is the unique map $N_y \times I \rightarrow E$

s.t. $p \circ \tilde{F} = F$, $\tilde{F}(y, 0) = f_0(y)$ on $N_y \times I$. $\square$