

§ 55 Applications of $\pi_1(S^1, *) \cong \mathbb{Z}$ Def. X : top. sp. , $A \subset X$

A is a retract of X if $\exists$ cont map $r: X \rightarrow A$
 s.t. $r(a) = a$ for $a \in A$; we say r : retraction

Example. $[0, 1] \subset \mathbb{R}$ $r(x) = \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases}$

Lem 55.1. $a \in A \subset X$, A retract of X Then $\pi_1(A, a)$ is a subgroup of $\pi_1(X, a)$ Proof write $j: A \rightarrow X$ for the inclusion map, r : retraction

we have $j_*: \pi_1(A, a) \rightarrow \pi_1(X, a)$
 $r_*: \pi_1(X, a) \rightarrow \pi_1(A, a)$ } homs

 r : retr. means $r \circ j = \text{id}_A$ $\Rightarrow r_* \circ j_* = (\text{id}_A)_* = \text{id}_{\pi_1(A, a)} \Rightarrow j_*$ is injective. $\square$ Thm 55.2 S^1 is not a retr. of $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ Proof $*$ = $(1, 0) \in S^1 \subset B^2$.

- $\pi_1(B^2, *) \cong \{0\}$: B^2 is contractible.
- $\pi_1(S^1, *) \cong \mathbb{Z}$ not a subgroup of $\{0\}$.

 $\Rightarrow$ conclusion $\square$

Lem.

Lem 55.3. $h: S^1 \rightarrow X$ cont. map. TFAE

1. h is nullhomotopic : homotopic to a const map
2. h extends to a cont map $k: B^2 \rightarrow X$
3. $h_*: \pi_1(S^1, *) \rightarrow \pi_1(X, h(x))$ is the triv. hom.



Proof of $1 \Rightarrow 2$

Take a homotopy $F: S^1 \times I \rightarrow X$ between h and a const. map $F(p, 1) = h(p)$, $F(p, 0) = x_0$.

Want: a map $k: B^2 \rightarrow X$

Step 1. $B^2 \cong (S^1 \times I) / \sim$ for $(p, t) \sim (p', t')$

$$\Leftrightarrow (p, t) = (p', t') \text{ or } t = 0 = t'$$

$$(S^1 \times I) / \sim \xrightarrow{\text{cpt}} B^2, [p, t] \mapsto tp. \quad \begin{array}{l} \text{cont, bijective.} \\ \text{Hausdorff} \end{array} \Rightarrow \text{homeo.}$$

Step 2. F gives k

$$(p, t) \sim (p', t') \Rightarrow F(p, t) = F(p', t')$$

$\Rightarrow F$ induces a cont. map $(S^1 \times I) / \sim \rightarrow X$

Proof of $2 \Rightarrow 3$: given ext $k: B^2 \rightarrow X$

$$\text{want } h_*([f]) = [e_{h(*)}]$$

$$k \text{ extends } h: h = k \circ j \quad \Rightarrow h_* = k_* \circ j_* \quad (j: S^1 \rightarrow B^2 \text{ incl.})$$

$$k_*: \pi_1(B^2, *) \rightarrow \pi_1(X, k(*)) \text{ is triv.}$$

$\uparrow$ triv. grp.

Proof of $3 \Rightarrow 1$ write $f_1: I \rightarrow S^1, x \mapsto (\cos 2\pi x, \sin 2\pi x)$
std parametrization

$$([f_1]) \in \pi_1(S^1, *) \text{ corresp. to } 1 \in \mathbb{Z}$$

$$h_*([f_1]) = [e_{h(*)}] \text{ means } h \circ f_1: I \rightarrow X \text{ is}$$

path homotopic to the const map $I \rightarrow X, x \mapsto h(*)$

$$\text{i.e. } \exists F: I \times I \rightarrow X \quad \begin{array}{l} F(s, 0) = h \circ f_1(s), F(s, 1) = h(*) \\ F(0, t) = h(*) = F(1, t) \end{array}$$

Step 1. consider $\sim$ on $I \times I$ by $(0, t) \sim (1, t)$

$$\text{then } (I \times I) / \sim \cong S^1 \times I$$



Step 2. homotop. between h and const map.

$$F \text{ induces } F': S^1 \times I \rightarrow X$$

$\square$

Cor 55.4 the incl. map $j: S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$ is not null-homotopic.

Proof. $S^1 \subset \mathbb{R}^2 \setminus \{0\}$ is a retract

$$r: \mathbb{R}^2 \setminus \{0\} \rightarrow S^1, v \mapsto \frac{v}{|v|}$$

$\Rightarrow$ Lem 55.1 $\pi_1(S^1, *) \cong \mathbb{Z}$ embeds into $\pi_1(\mathbb{R}^2 \setminus \{0\}, *)$ by j_*

$\Rightarrow$ conclusion
Lem 55.3

Def. vector field on $X \subset \mathbb{R}^2$: cont map $v: X \rightarrow \mathbb{R}^2$
interpret as cont. distr. of vectors



Th'm 55.5 v : non-vanishing vec. field on B^2 .
($v(x) \neq 0$ for $x \in B^2$)

Then $\exists p, q$ s.t. $\begin{cases} v(p) \text{ points outward} \\ v(q) \text{ points inward} \end{cases}$



$(v(p), p) > 0$ for inn. prod
"normal vec"

$$(v(q), q) < 0$$

Proof by contradiction. suppose no such q exists.

v is a map $B^2 \rightarrow \mathbb{R}^2 \setminus \{0\}$ by assumption

Step 1 $w = v|_{S^1}: S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$ is null homotop.

Lem 55.3

Step 2. w is homotop to the incl map $j: S^1 \rightarrow \mathbb{R}^2 \setminus \{0\}$

$$F: S^1 \times I \rightarrow \mathbb{R}^2 \setminus \{0\}, F(p, t) = tp + (1-t)w(p)$$

$$F(p, t) \neq 0 \text{ by } (p, w(p)) > 0. (\Rightarrow (p, F(p, t)) > 0)$$

Step 3. contradiction: $j \simeq_h \text{const. map}$

$j \simeq_h w \simeq_h \text{const.}$ contradicts Cor 55.4

"no such p " $\Rightarrow$ contr. by considering $-v$.

□

Brouwer's fixed point thm (55.6) $f: B^2 \rightarrow B^2$ cont.
 then $\exists p \in B^2$ s.t. $f(p) = p$.

Proof by contr. suppose no such p exists.

Put $v(p) = f(p) - p$ ($\neq 0$ by assumption.)

Step 1. $p \in S^1 \Rightarrow v(p)$ points inward

$$f(p) \in B^2 \Rightarrow |(p, f(p))| < 1$$

$$f(p) \neq p$$

$$\Rightarrow (p, v(p)) < 1 - (p, p) = 0$$

Step 2. contradiction with Thm 55.5 $\square$