

§§ 31, 32 Normal spaces

We saw two "separation" conditions

- Hausdorff spaces : $\forall x \neq y \in X \exists U, V$ open sets of X
 $U \cap V = \emptyset, x \in U, y \in V$



- T_1 spaces : $\forall x \in X : \{x\}$ is closed
 $(\Leftrightarrow \forall x \neq y \in X \exists V$ open set of $X : x \notin V \ni y)$

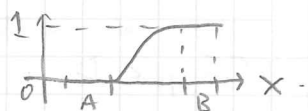
Def. X : top. sp. ; X is normal if $\forall A, B \subset X$ closed,
 $A \cap B = \emptyset \exists \bar{U}, \bar{V} \subset X$ open $A \subset \bar{U}, B \subset \bar{V}, \bar{U} \cap \bar{V} = \emptyset$



Rem T_1 & normal $\Rightarrow$ Hausdorff

Why is this important?

- compact Hausdorff $\Rightarrow$ normal (Thm 32.3)
- Urysohn's lemma (Thm 33.1) X normal, A, B closed
 $A \cap B = \emptyset \Rightarrow \exists f : X \rightarrow \mathbb{R}$ cont. $f|_A = 0, f|_B = 1$.



separate closed sets by
 cont. real funcs.

so X cpt Hausdorff $\Rightarrow C(X, \mathbb{R}) = \{f : X \xrightarrow{f} \mathbb{R} \text{ cont.}\}$
 has enough elements

$(\Rightarrow M = \{\varphi : C(X, \mathbb{R}) \rightarrow \mathbb{R} \text{ ring hom}\}$ is homeo. to X)
 "Gelfand-Naimark duality"

Thm 32.3 X cpt Haus. $\Rightarrow X$ normal

Proof $A, B \subset X$ closed, $A \cap B = \emptyset$

Want $U, V \subset X$ open $A \subset U, B \subset V, \bar{U} \cap \bar{V} = \emptyset$

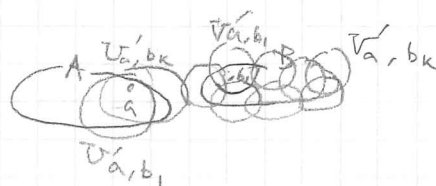
Step 1 $\forall a \in A \exists U_a, V_a$ open $a \in U_a, B \subset V_a$
 $(X \text{ is regular})$

Haus. $\Rightarrow \forall b \in B \exists U'_{a,b}, V'_{a,b}$ open $a \in U'_{a,b}, b \in V'_{a,b}$

cont.) $B \subset \bigcup_{b \in B} V'_{a,b}$, B compact

$$\Rightarrow \exists b_1, \dots, b_k \quad B \subset \bigcup_{i=1}^k V'_{a,b_i} = \bar{V}_a$$

$$U = \bigcap_{i=1}^k V'_{a,b_i}$$



Step 2 claim

$$A \text{ compact}, A \subset \bigcup_{a \in A} U_a \Rightarrow \exists a_1, \dots, a_n : A \subset \bigcup_{i=1}^n U_{a_i}$$

$$V = \bigcap_{i=1}^n U_{a_i}$$

§ 33 Urysohn's lemma

Th'm 33.1 X normal, $A, B \subset X$ closed, $A \cap B = \emptyset$

Then $\exists$ cont. $f: X \rightarrow \mathbb{R}$, $f(a) = 0$ ($a \in A$)
 $f(b) = 1$ ($b \in B$)

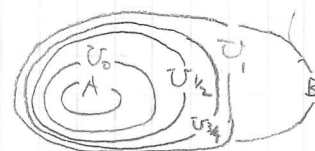
Proof. Step 1. $D = \left\{ \frac{n}{2^k} : 0 \leq n \leq 2^k \right\}$ dyadic rationals

$\exists$ open sets $U_p \subset X$ ($p \in D$) s.t.

$$- p < q \Rightarrow \overline{U_p} \subset U_q$$

closure.

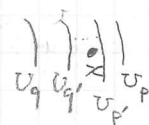
$$- A \subset U_0, U_1 = X \setminus B$$



Step 2 put $f(x) = \begin{cases} \inf \{ p \in D : x \in U_p \} & x \notin B \\ 1 & x \in B \end{cases}$

$\Rightarrow f$ is cont.

$$f|_A = 0, f|_B = 1.$$



$$q, q' < f(x)$$

$$p, p' > f(x)$$

Proof of Step 1. induction on k for $p = \frac{n}{2^k}$

$k=0$; U_0 : open set s.t. $A \subset U_0$, $\exists V$ open $B \subset V$
 (possible by normality) $U_0 \cap V = \emptyset$

$$U_1 = X \setminus B$$

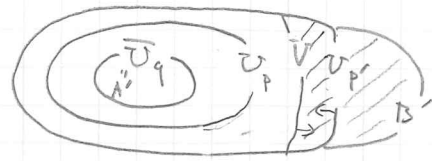
after having U_p for $p = \frac{n}{2^k}$ s.t. $q < p \Rightarrow \overline{U_q} \subset U_p$

define U_p for $p = \frac{n}{2^{k+1}}$, n odd by:

$$A' = \overline{U_q} \quad q = \frac{n-1}{2^{k+1}}, \quad B' = X \setminus U_{q'}, \quad q' = \frac{n+1}{2^{k+1}}$$

cont. U_p : open, $A' \subset U_p$, $\exists V$ s.t. $B' \subset V$
 $U_p \cap V = \emptyset$

$$\Rightarrow \overline{U_p} \subset X \setminus V \subset U_{q'}$$



Proof of Step 2

observe $x \in \overline{U_p} \Rightarrow f(x) \leq p$, $x \notin U_p \Rightarrow f(x) \geq p$.

To check the continuity of f , it's enough to

show $f^{-1}((s, t))$ is open for $s, t \in \mathbb{R}$.

↑ open int.; they form a base of top. for $\mathbb{R}$.

$\Rightarrow$ given $x \in f^{-1}((s, t))$, want an open set $W \subset X$ s.t.

$$x \in W \subset f^{-1}((s, t))$$

take $p, q \in \mathbb{D}$ s.t. $s < q < f(x) < p < t$

(if $f(x) = 0$, only take p , if $f(x) = 1$, only take q)

$W = U_p \cap (X \setminus \overline{U_q})$ is open, contains x , and

$$y \in W \Rightarrow q \leq y \leq p \quad \square$$

Another separation condition : soberness.

"lattice structure of open sets can determine points"

$\forall f: \mathcal{T} \rightarrow \{0, 1\}$ s.t.

$$- B \subset \mathcal{T} \text{ subcoll} \quad f\left(\bigcup_{U \in B} U\right) = \max_{U \in B} f(U)$$

$$- B \subset \mathcal{T} \text{ fin. subcoll} \quad f\left(\bigcap_{U \in B} U\right) = \min_{U \in B} f(U)$$

$$\exists! x \in X \text{ s.t. } f(U) = \begin{cases} 1 & x \in U \\ 0 & x \notin U \end{cases}$$

Ex. R commutative ring

(Mandatory assign.)

$$\Rightarrow \text{Spec}(R) = \{P \subset R : \text{prime ideal}\}$$

$$\text{with closed sets } \{V(I) = \{P : I \subset P\} \subset \text{Spec}(R) \mid I \in R\}$$

is sober.