

§ 46 Compact open topology.

$$Y^X = \{ f: Y \rightarrow X \text{ (any) map} \} \simeq \prod_{x \in X} Y$$

$$C(Y, X) = \{ f: Y \rightarrow X \text{ cont. map} \} \subset Y^X$$

Recall: top. of pointwise conv. on Y^X ;

$$\text{from } B_0 = \{ S(x, U) = \{ f \in Y^X : f(x) \in U \} : \\ x \in X, U \subset Y \text{ open} \}$$

$$\text{and } B = \{ S_1 \cap \dots \cap S_k : k = 0, 1, \dots; S_i \in B_0 \} \\ = \{ \{ f \in Y^X : f(x_i) \in U_i \} \\ x_1, \dots, x_k \in X, U_1, \dots, U_k \subset Y \text{ open} \} \\ \text{as base.}$$

(identified with the prod. top. on $\prod_{x \in X} Y$)

Sequence $(f_i)_{i=1}^{\infty}$ conv. to f in this top.

$$\Leftrightarrow \forall x \in X : f_i(x) \rightarrow f(x) \quad (i \rightarrow \infty)$$

Variation: replace (fin. coll of) points by compact subsets of X

Def. the compact-open topology on $C(X, Y)$ is defined from

$$B_0 = \{ S(C, U) = \{ f \in C(X, Y) : f(C) \subset U \} : \\ C \subset X \text{ cpt}, U \subset Y \text{ open} \}$$

$$B = \{ S_1 \cap \dots \cap S_k : S_i \in B_0 \} \text{ as base}$$

Rem 1, when Y is a met. sp., this top. is also called the top. of compact convergence neighborhoods of f :

$$B_C(f, \varepsilon) = \{ g : \forall x \in C : \underset{\substack{\uparrow \\ \text{dist. on } Y}}{d(f(x), g(x))} < \varepsilon \}$$

for $C \subset X$ cpt., $\varepsilon > 0$

2. this top. makes sense on Y^X , $C(X, Y)$ closed in Y^X

(cont.) hence $(f_i)_{i=1}^{\infty} \in \mathcal{C}(X, Y)$, $f_i \rightarrow f \in Y^X$
in this top. $\Rightarrow f$ cont.

3. (Thm 46.7)

top. of unif. conv. is finer than top. of cpt. conv.

top. of cpt. conv. is finer than top. of pointwise conv.

Thm 46.10 X : loc. cpt. Hausdorff, Y : top. sp
($\forall x \in X$ has a cpt. neighborhood)

The evaluation map $ev: X \times \mathcal{C}(X, Y) \rightarrow Y$, $(x, f) \mapsto f(x)$
is cont. with resp. to the cpt-open top. on $\mathcal{C}(X, Y)$

Proof. Take $U \subset Y$ open. Want: $ev^{-1}(U)$ open.

i.e. $\forall (x, f) \in ev^{-1}(U) \exists$ neigh. Z of (x, f)
s.t. $Z \subset ev^{-1}(U)$.

Step 0 $f \in \mathcal{C}(X, Y) \Rightarrow f^{-1}(U)$ is an open set of X .

$(x, f) \in ev^{-1}(U) \Rightarrow f(x) \in U$ i.e. $x \in f^{-1}(U)$.

$\Rightarrow \exists$ neigh. V of x s.t. $V \subset f^{-1}(U)$

Step 1 we may assume $\bar{V}$ is cpt., $f(\bar{V}) \subset U$

Take V_0 : open neigh. of x , $\bar{V}_0$ cpt.

$\bar{V}_0$ is cpt. Haus. $\Rightarrow$ normal.

$A = \{x\}$, $B = \bar{V}_0 \setminus V$ are nonintersecting closed sets.
 $\Rightarrow$ normal. $\exists$ open sets W, W' (induced top.)

s.t. $x \in W$, $B \subset W'$, $W \cap W' = \emptyset$

Find $W_0 \subset X$ open s.t. $W = W_0 \cap \bar{V}_0$; Put $W_1 = W_0 \cap V_0$

$x \in W_1$, $\bar{W}_1 \subset \bar{V}_0$ cpt., $\bar{W}_1 \subset \bar{W} \subset \bar{V}_0 \setminus B \subset V$
hence $f(\bar{W}) \subset U$.



Step 2 $Z = C \times V \times S(\overline{V}, U)$ works ; i.e.
 $(x, f) \in Z = \text{ev}^{-1}(U)$ $\xleftarrow{\text{cpt.}}$
 $\xleftarrow{\text{open for cpt-open top.}}$ $\square$

Th'm 46.11 $\rightarrow$ There is a bijective corresp.

$$\left\{ \begin{array}{l} X \times Z \xrightarrow{f} Y \text{ cont.} \\ X \text{ loc. cpt. Hausdorff} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} Z \xrightarrow{F} \mathcal{C}(X, Y) : \text{cont} \\ f(x, z) = F(z)(x) \end{array} \right\}$$

Proof. Step 1 $f \mapsto F$ is well-defined

$$X \times Z \xrightarrow{\text{id}_X \times F} X \times \mathcal{C}(X, Y) \xrightarrow{\text{ev}} Y \text{ is } f.$$

$\uparrow \text{cont. by continuity of } F$ $\uparrow \text{cont. by Th'm 46.10}$

Step 2 $f \mapsto F$ is well-defined

want. $F^{-1}(S(C, U))$ is open in Z
 for $C \subset X$ cpt. $U \subset Y$ open.

i.e. $\forall z \in F^{-1}(S(C, U)) \exists$ neigh. W of z s.t.
 $W \subset F^{-1}(S(C, U))$

assumption on z : $f(x, z) \in U$ for $x \in C$

for each x , $f_x = \left\{ \begin{array}{l} U_x \text{ neigh. of } x \\ W_x \text{ neigh. of } z \end{array} \right\}$ s.t. $U_x \times W_x \subset f^{-1}(U)$

By cpt-ness $\exists x_1, \dots, x_k$ s.t. $C \subset \bigcup_{i=1}^k U_{x_i}$

then $W = \bigcap_{i=1}^k W_{x_i}$ satisfies $C \times W \subset f^{-1}(U)$
 this W will do.

Examples 1. $Z = I$. homotopy of cont maps as
 $I \rightarrow \mathcal{C}(X, Y)$

2. free loop space $\Omega X = \mathcal{C}(S^1, X)$.

3. $x \in X$; loop space $\Omega(X, x) = \{f: S^1 \rightarrow X, f(x) = x\}$
 with cpt-open top.

(we often write ΩX)

path-homotop. of loops at $x \leftrightarrow$ cont maps $I \rightarrow \Omega(X, x)$