

**SAMPLE SOLUTIONS FOR MAT3500/MAT4500
FINAL EXAM, AUTUMN 2023**

PROBLEM 1

(a) ... subsets of X such that the union of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$, and the intersection of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$. In particular, the collection $\mathcal{T}$ must contain $\emptyset$ and X .

(b) ... the union of all open subsets of X that are contained in A .

(c) ... it has a countable basis at each point $x \in X$, i.e., for each $x \in X$ there is a countable collection $\mathcal{B}_x$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}_x$.

(d) ... the inverse image $p^{-1}(U)$ can be written as the union of pairwise disjoint open sets V_α in E such that for each α the restriction of p to V_α is a homeomorphism of V_α onto U .

PROBLEM 2

(a) The first projection $\pi_1: X \times X \rightarrow X$ is continuous and surjective, since for each $x \in X$ we have $\pi_1(x, x) = x$. Hence X is the continuous image of the connected space $X \times X$, and must therefore be connected by Theorem 23.5. If X were finite then $X \times X$ would be finite by Corollary 6.8, contradicting the assumption that $X \times X$ is infinite. Hence X is infinite.

(b) (We closely follow the proof of Theorem 23.6.) We can choose a point $a \in X - F$, since X is infinite by (a) and F is finite. For each $x \in X$ the union $C_x = X \times \{a\} \cup \{x\} \times X$ is connected by Theorem 23.3, since $X \times \{a\} \cong X$ and $\{x\} \times X$ are both connected, and have the point (x, a) in common. Hence the union

$$\bigcup_{x \in X - F} C_x = (X \times \{a\}) \cup ((X - F) \times X)$$

is connected, since these connected subspaces all have the point (a, a) in common. Interchanging the roles of the coordinates, it also follows that

$$(\{a\} \times X) \cup (X \times (X - F))$$

is connected. The two displayed subspaces both contain (a, a) , and therefore their union

$$((X - F) \times X) \cup (X \times (X - F)) = X \times X - F \times F$$

is also connected. Here we use that $X \times \{a\} \subset X \times (X - F)$ and $\{a\} \times X \subset (X - F) \times X$, so these subsets do not add any points to the union.

(c) (This is a variant of Exercise 24.1.) No, this is not possible. By (a), such an X would be connected and infinite. Let $F = \{b, c\} \subset X$ be any subset with two elements. If there existed a homeomorphism

$$h: X \times X \xrightarrow{\cong} S^1,$$

then h would restrict to a homeomorphism

$$h': X \times X - F \times F \xrightarrow{\cong} S^1 - h(F \times F)$$

i.e., S^1 minus four different points. This would be homeomorphic to a disjoint union of four open intervals, and cannot be connected. This would contradict (b).

Alternatively: If $S^1 \simeq X \times X$ then $\mathbb{Z} \cong \pi_1(S^1) \cong \pi_1(X \times X) \cong \pi_1(X) \times \pi_1(X)$, for appropriate choices of base points, but there is no group π with $\mathbb{Z} \cong \pi \times \pi$. This proves that there cannot be a homotopy equivalence $X \times X \simeq S^1$.

PROBLEM 3

(a) (This is similar to Theorem 23.5, with path-connected in place of connected.) Let $f: X \rightarrow Y$ be a continuous surjection, with X path-connected. For any two points $y_0, y_1 \in Y$ there exist points $x_0, x_1 \in X$ with $f(x_0) = y_0$ and $f(x_1) = y_1$, since f is surjective. Since X is path-connected, there exists a path $\alpha: [0, 1] \rightarrow X$ in X from $\alpha(0) = x_0$ to $\alpha(1) = x_1$. Then $f \circ \alpha: [0, 1] \rightarrow Y$ is a path in Y from $(f \circ \alpha)(0) = f(x_0) = y_0$ to $(f \circ \alpha)(1) = f(x_1) = y_1$, by Theorem 18.2(c). Hence Y is path-connected.

(b) (This is similar to the hint for Exercise 25.8.) Let D be a path-component of B . For each path-component C of A , the image $q(C)$ is a path-connected subspace of B , and is therefore either contained in D or disjoint from D , by Theorem 25.2. Equivalently, either $C \subset q^{-1}(D)$ or $C \cap q^{-1}(D) = \emptyset$. Hence $q^{-1}(D)$ equals the union of the path components C in B with $C \subset q^{-1}(D)$. By assumption, each such C is open, hence so is their union. This proves that $q^{-1}(D)$ is open. Since q is a quotient map, this implies that D is open.

(c) (This is similar to the rest of Exercise 25.8.) Let $y \in Y$ be any point, and let B be any neighborhood of y in Y . Let $A = p^{-1}(B)$, and define a surjective map $q: A \rightarrow B$ by restricting the source and target (= domain and codomain) of p to A and B , respectively.

$$\begin{array}{ccccc} A & \subset & X \\ \downarrow q & & \downarrow p \\ y \in D \subset B & \subset & Y \end{array}$$

Then A is open in X , since B is open in Y and p is continuous, so q is a quotient map by Theorem 22.1. Moreover, each path component of A is open in A , since X is locally path-connected, by Theorem 25.3. Let D be the path component of y in B . Part (b) shows that D is open in B , and therefore also open in Y , since B is open in Y , using Lemma 16.2. This confirms that

y has a path-connected neighborhood D contained in the arbitrarily chosen neighborhood B , so Y is locally path-connected.

PROBLEM 4

(a) (This closely follows the proof of Theorem 34.1.) The compact Hausdorff space X is normal, by Theorem 32.3. Let $K \subset J \times J$ be the subset of pairs $\gamma = (\alpha, \beta)$ for which $\bar{B}_\alpha \subset B_\beta$. By Urysohn's lemma (Theorem 33.1) applied to the disjoint closed subsets $\bar{B}_\alpha$ and $X - B_\beta$ of X , we can choose a map $g_{\alpha,\beta}: X \rightarrow [0, 1]$ with $g_{\alpha,\beta}(\bar{B}_\alpha) \subset \{1\}$ and $g_{\alpha,\beta}(X - B_\beta) \subset \{0\}$.

Then $\{g_\gamma\}_{\gamma \in K}$ has the required property. To see this, let $x \in U \subset X$ with U open. Since $\mathcal{B}$ is a basis for the topology on X , there exists a $\beta \in J$ with $x \in B_\beta \subset U$. Since X is normal (hence regular), there exists a neighborhood V of x with $\bar{V} \subset B_\beta$, by Lemma 31.1(a). Again, since $\mathcal{B}$ is a basis, there exists an $\alpha \in J$ with $x \in B_\alpha \subset V$. Then $\bar{B}_\alpha \subset \bar{V} \subset B_\beta$, as in Exercise 17.6(a), so $\gamma = (\alpha, \beta) \in K$. Moreover, $x \in B_\alpha \subset \bar{B}_\alpha$, so $g_\gamma(x) = 1$, and $X - U \subset X - B_\beta$, so $g_\gamma(X - U) \subset g_\gamma(X - B_\beta) \subset \{0\}$, as required.

(b) (This closely follows the proof of Theorem 34.2.) Let $G: X \rightarrow [0, 1]^K$ be given by $G(x) = (g_\gamma(x))_{\gamma \in K}$ for $x \in X$. Then G is continuous, since $[0, 1]^K$ has the product topology and each component $\pi_\gamma \circ G = g_\gamma: X \rightarrow [0, 1]$ is continuous. Moreover, G is injective, since if $x \neq y \in X$ then using (a) with $U = X - \{y\}$ there exists a $\gamma \in K$ with $g_\gamma(x) = 1$ and $g_\gamma(y) = 0$. Hence $\pi_\gamma G(x) \neq \pi_\gamma G(y)$, so that $G(x) \neq G(y)$. Since X is compact and $[0, 1]^K$ is Hausdorff (by Theorem 19.4), it follows that G is an embedding (as in Theorem 26.6).