

MAT3500/MAT4500 FALL 2024
SAMPLE SOLUTIONS FOR MANDATORY ASSIGNMENT

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PROBLEM 1

Let (X, d) be a metric space. The ϵ -balls

$$B_d(p, \epsilon) = \{q \in X \mid d(p, q) < \epsilon\}$$

for varying $p \in X$ and $\epsilon > 0$ give a basis for the metric topology $\mathcal{T}_d$. Fix one $p \in X$. Then each $U_n := B_d(p, 1/n)$ with $n \in \mathbb{N}$ is a basis element for the topology, hence is open, and contains p , hence is a neighborhood of p . Moreover, each neighborhood V of p is an open set containing p , so there is an $\epsilon > 0$ with $B_d(p, \epsilon) \subset V$. Choosing $n \in \mathbb{N}$ so that $1/n < \epsilon$, it follows that $U_n \subset B_d(p, \epsilon) \subset V$, so $\{U_n\}_{n \in \mathbb{N}}$ is a neighborhood basis at p . Since the indexing set $\mathbb{N}$ is countable, this is a countable neighborhood basis. Since $1/(n+1) < 1/n$ for each $n \in \mathbb{N}$, we have $U_{n+1} = B_d(p, 1/(n+1)) \subset B_d(p, 1/n) = U_n$, so this is a sequential neighborhood basis.

PROBLEM 2

For each $n \in \mathbb{N}$ let $V_n = U_1 \cap U_2 \cap \cdots \cap U_n$. This is a finite intersection of open sets containing p , hence is an open set containing p , i.e., is a neighborhood of p . Moreover,

$$V_{n+1} = V_n \cap U_{n+1} \subset V_n,$$

so this is a sequential neighborhood basis.

PROBLEM 3

For each $n \in \mathbb{N}$, let $W_n = U_n \cap V_n$. This is an intersection of two open sets containing p , hence is a neighborhood of p . Moreover $W_n = U_n \cap V_n \subset U_n$ and $W_n = U_n \cap V_n \subset V_n$ for each $n \in \mathbb{N}$, as required. Each neighborhood of p contains U_n for some n , hence also contains W_n for the same n , so $\{W_n\}_{n \in \mathbb{N}}$ is a neighborhood basis of 0. By assumption, $U_{n+1} \subset U_n$ and $V_{n+1} \subset V_n$ for each $n \in \mathbb{N}$, so

$$W_{n+1} = U_{n+1} \cap V_{n+1} \subset U_n \cap V_n = W_n,$$

proving that $\{W_n\}_{n \in \mathbb{N}}$ is a sequential neighborhood basis at p .

PROBLEM 4

Let $\mathcal{B}_p = \{U_n\}_{n \in \mathbb{N}}$. We define $\phi(n)$ by induction on $n \in \mathbb{N}$. For $n = 1$, note that W_1 is a neighborhood of p , so there exists some element U_i in $\mathcal{B}_p$ with $p \in U_i \subset W_1$. Let $\phi(1) = i \in \mathbb{N}$. For $n \geq 2$, suppose that $\phi(n-1) \in \mathbb{N}$ has already been chosen. Since W_n is a neighborhood of p , there exists an element U_j in $\mathcal{B}_p$ with $p \in U_j \subset W_n$. Let $\phi(n) = \max\{j, \phi(n-1) + 1\}$. Then $\phi(n) \geq j$, so $U_{\phi(n)} \subset U_j \subset W_n$, and $\phi(n) \geq \phi(n-1) + 1$. The second condition ensures that the function $\phi: \mathbb{N} \rightarrow \mathbb{N}$ defined this way is strictly increasing.

PROBLEM 5

Each neighborhood V of 0 is an open set containing 0, so there exists an element B in the basis for $\mathcal{T}^C$ with $0 \in B \subset V$. Each basis element is a finite intersection

$$B = \pi_{c_1}^{-1}((a_1, b_1)) \cap \cdots \cap \pi_{c_t}^{-1}((a_t, b_t))$$

of subbasis elements, where $t \geq 0$ and $c_s \in C$ and $a_s < b_s \in \mathbb{R}$ for each $1 \leq s \leq t$. Since $0 \in B$ we must have $0 \in \pi_{c_s}^{-1}((a_s, b_s))$ for each s , which means that $0 = \pi_{c_s}(0) \in (a_s, b_s) \subset \mathbb{R}$ for each s , so that $a_s < 0 < b_s$, as claimed.

PROBLEM 6

For each finite subset $F = \{c_1, \dots, c_t\} \subset C$ and natural number m ,

$$U(F, 1/m) = \pi_{c_1}^{-1}((-1/m, 1/m)) \cap \cdots \cap \pi_{c_t}^{-1}((-1/m, 1/m))$$

is a basis element for $\mathcal{T}^C$ that contains 0, hence is a neighborhood of 0. By Problem 5, each neighborhood V of 0 contains a basis element

$$B = \pi_{c_1}^{-1}((a_1, b_1)) \cap \cdots \cap \pi_{c_t}^{-1}((a_t, b_t))$$

with $a_s < 0 < b_s$ for each $1 \leq s \leq t$. Choose $m \in \mathbb{N}$ so large that $1/m \leq -a_s$ and $1/m \leq b_s$ for each $1 \leq s \leq t$, and let $F = \{c_1, \dots, c_t\}$, which is a finite subset of C . Then

$$0 \in U(F, 1/m) \subset B \subset V.$$

Hence the collection $\mathcal{B}_0$ of $U(F, 1/m)$ with $F \subset C$ finite and $m \in \mathbb{N}$ is a neighborhood basis of 0.

PROBLEM 7

The product of any finite number of countable sets is countable, and the image of a countable set under any function is again countable. Hence the image

$$C_t := \{F \mid F \subset C \text{ has at most } t \text{ elements}\}$$

of the function

$$\prod_{s=1}^t C \longrightarrow \{F \mid F \subset C \text{ is finite}\}$$

$$(c_1, \dots, c_t) \longmapsto \{c_1, \dots, c_t\}$$

is countable, for each integer $t \geq 0$. Moreover, a countable union of countable sets is countable, so

$$\{F \mid F \subset C \text{ is finite}\} = \bigcup_{t \geq 0} C_t$$

is countable.

PROBLEM 8

Yes. The indexing set $\{F \mid F \subset C \text{ is finite}\}$ for the neighborhood basis $\mathcal{B}_0$ is countable by Problem 7, so this is a countable neighborhood basis of 0.

PROBLEM 9

The product topology $\mathcal{T}^D$ has a subbasis given by the collection

$$\{\pi_d^{-1}((a, b)) \mid d \in D, a < b \in \mathbb{R}\}$$

with elements

$$\pi_d^{-1}((a, b)) = \{g: D \rightarrow \mathbb{R} \mid a < g(d) < b\}$$

and a basis given by the collection of finite intersections

$$B = \pi_{d_1}^{-1}((a_1, b_1)) \cap \cdots \cap \pi_{d_t}^{-1}((a_t, b_t))$$

with $d_s \in D$ and $a_s < b_s$ for each $1 \leq s \leq t$. For each n , the neighborhood U_n of 0 must contain one of these basis elements B , with $a_s < 0 < b_s$ in each case. Letting $G_n = \{d_1, \dots, d_t\}$, and choosing $m_n \in \mathbb{N}$ with $1/m_n \leq -a_s$ and $1/m_n \leq b_s$ for each $1 \leq s \leq t$, we have

$$V(G_n, 1/m_n) \subset B \subset U_n.$$

This holds, because $|g(d)| < 1/m_n$ for each $d \in G_n$ implies $a_s < g(d_s) < b_s$ for each $1 \leq s \leq t$.

PROBLEM 10

The set $\bigcup_{n \in \mathbb{N}} G_n$ is a countable union of finite subsets of D , hence is a countable subset of D . It cannot be equal to D , since this set is by assumption uncountable. Hence it is a proper subset of D , and we can let e be any element of $D \setminus \bigcup_{n \in \mathbb{N}} G_n$.

PROBLEM 11

Consider any $n \in \mathbb{N}$. By Problem 9 we have $V(G_n, 1/m_n) \subset U_n$. If also $U_n \subset V(\{e\}, 1)$, we would have $V(G_n, 1/m_n) \subset V(\{e\}, 1)$, i.e., that any function $g: D \rightarrow \mathbb{R}$ with $|g(d)| < 1$ for $d \in G_n$ satisfies $|g(e)| < 1$. But $e \notin G_n$, so the “indicator” or “characteristic” function

$$g(x) = \begin{cases} 0 & \text{for } x \neq e, \\ 1 & \text{for } x = e \end{cases}$$

is a counterexample to this statement. Hence $V(\{e\}, 1)$ does not contain U_n for any n .

PROBLEM 12

No. If a countable neighborhood basis $\{U_n\}_{n \in \mathbb{N}}$ of 0 existed, then Problem 11 would exhibit a neighborhood $V(\{e\}, 1)$ of 0 that contradicts the defining property of $\{U_n\}_{n \in \mathbb{N}}$ being a neighborhood basis.

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