

**SAMPLE SOLUTIONS FOR MAT3500/MAT4500
FINAL EXAM, AUTUMN 2024**

PROBLEM 1

- (a) ... for each open subset V of Y the preimage $f^{-1}(V)$ is open in X .
- (b) ... there does not exist a pair U, V of nonempty, open subsets of X with $U \cap V = \emptyset$ and $U \cup V = X$.
- (c) ... a second-countable Hausdorff space X such that each point in X has a neighborhood that is homeomorphic to (an open subset of) $\mathbb{R}^m$.
- (d) ... a path homotopy class of loops in X based at x_0 .

PROBLEM 2

(a) Each $a \in \mathbb{Z}$ lies in $U(1, 0) \in \mathcal{B}$. Consider any $B_1, B_2 \in \mathcal{B}$ and $a \in B_1 \cap B_2$. Then $B_1 = U(m, c)$ for some integers $m \geq 1$ and c , and since $a \in B_1$ we must have $B_1 = U(m, a)$ for the same m . Similarly, $B_2 = U(n, a)$ for some $n \geq 1$. Let $B_3 = U(mn, a)$. Then $B_3 \in \mathcal{B}$ and $a \in B_3 \subset B_1 \cap B_2$. Hence $\mathcal{B}$ satisfies the condition for being a basis for a topology.

(b) Remainder after division by m partitions the integers as a disjoint union

$$\mathbb{Z} = \coprod_{r=0}^{m-1} U(m, r)$$

so that

$$\mathbb{Z} - U(m, c) = \coprod_{\substack{0 \leq r < m \\ r \neq c}} U(m, r).$$

Each basis element is open, hence the right-hand union of basis elements is open, and its complement, namely $U(m, c)$, must be closed.

(c) To show that μ is continuous at (a, b) , it suffices to prove that $\mu^{-1}(B)$ contains a neighborhood of (a, b) for each basis element $B \subset X$ that contains $\mu(a, b) = ab$. Here $B = U(m, ab)$ for some $m \geq 1$. The product $W = U(m, a) \times U(m, b)$ is a basis element for the product topology on $X \times X$, and $(a, b) \in W$. We claim that $W \subset \mu^{-1}(B)$. A general element of W has the form $(mx + a, my + b)$ with $x, y \in \mathbb{Z}$, and $\mu(mx + a, my + b) = (mx + a)(my + b) = m(x + y + mxy) + ab$ is an element of $U(m, ab) = B$. This confirms the claim. Hence W is the required neighborhood.

(d) Yes, X is Hausdorff. Given $a, b \in X$ with $a \neq b$, choose $m > |b - a|$. Then $a \in U = U(m, a)$ and $b \in V = U(m, b)$. These basis elements U and V are open. Moreover, $U \cap V = \emptyset$, since if $z \in U \cap V$ then $z = mx + a$ and

$z = my + b$ for some $x, y \in \mathbb{Z}$, so that $b - a = m(x - y)$ is a multiple of m . Since $|b - a| < m$, we must have $b - a = 0$, but we assumed $a \neq b$.

(e) For each $m \geq 1$ the discrete topology on Y_m is associated to the (discrete, bounded) metric $d_m = \bar{d}_m$ given by

$$d_m(p_m, q_m) = \begin{cases} 0 & \text{if } p_m = q_m, \\ 1 & \text{otherwise,} \end{cases}$$

for $p_m, q_m \in Y_m$. We proved in the course that any countable product $Y = \prod_m Y_m$ of metrizable spaces is itself metrizable, e.g. with the metric

$$D(p, q) = \sup_{m \geq 1} \frac{d_m(p_m, q_m)}{m}$$

for $p = (p_m)_{m \geq 1}, q = (q_m)_{m \geq 1} \in Y$.

Alternative argument: Discrete spaces are regular, and products of regular spaces are regular, so Y is regular. Finite spaces are second-countable, and countable products of second-countable spaces are second-countable, so Y is second-countable. Hence Y is metrizable by Urysohn's metrization theorem.

(f) The function $r: X \rightarrow Y$ is injective, since given $a, b \in X$ with $a \neq b$ we can choose $m > |b - a|$, and then $r_m(a) \neq r_m(b)$ in Y_m , which implies that $r(a) \neq r(b)$ in Y . Hence r corestricts to a bijection $s: X \rightarrow r(X)$, which we must show is a homeomorphism when $r(X)$ has the subspace topology from Y .

$$X \xrightarrow{s} r(X) \subset Y \xrightarrow{\pi_m} Y_m$$

For each $m \geq 1$ the singleton sets $\{\{c\} \mid 0 \leq c < m\}$ give a (sub-)basis for the discrete topology on Y_m , so the preimages

$$\pi_m^{-1}(\{c\})$$

for the projections $\pi_m: Y \rightarrow Y_m$ give the standard subbasis for the product topology on Y , and the intersections

$$r(X) \cap \pi_m^{-1}(\{c\})$$

for $0 \leq c < m$ give a subbasis for the subspace topology on $r(X)$. To check that the bijection s is a homeomorphism, we must verify that the preimages

$$s^{-1}(r(X) \cap \pi_m^{-1}(\{c\})) = r^{-1}(\pi_m^{-1}(\{c\}))$$

for $0 \leq c < m$ generate the given topology $\mathcal{T}$ on X . However,

$$r^{-1}(\pi_m^{-1}(\{c\})) = \{a \in X \mid r_m(a) = c\} = U(m, c),$$

so this collection of preimages is equal to $\mathcal{B}$, hence generates precisely the topology $\mathcal{T}$.

(g) Yes, by (e) and (f) the space X is homeomorphic to a subspace of a metrizable space, hence is metrizable, and is therefore normal.

Alternatively, one can first prove that X is regular, by noting that for each $a \in W \subset X$ with W open, there is an $m \geq 1$ with $a \in U(m, a) \subset W$, and then using (b) we see that $U = U(m, a)$ satisfies $a \in U \subset \bar{U} \subset W$. Together with the positive answer in (d), this proves regularity. Since $\mathcal{B}$ in (a) gives a countable basis for the topology on X , proving second-countability, it follows by a theorem of Tychonoff that X is normal.

(h) We can partition X as a disjoint union of nonempty open subsets

$$X = \coprod_{k \geq 1} U(2^k, c_k),$$

for suitably chosen integers $(c_k)_{k \geq 1}$. Then

$$\mathcal{C} = \{U(2^k, c_k)\}_{k \geq 1}$$

is an open cover of X that does not admit any finite subcover, proving that X is not compact. The integers c_k may be chosen inductively for $k \geq 1$ by letting c_k be an element of minimal absolute value in

$$X - \coprod_{i=1}^{k-1} U(2^i, c_i).$$

One such integer sequence is given by

$$c_k = \sum_{i=0}^{k-2} (-2)^i = \frac{1 - (-2)^{k-1}}{3}.$$

By induction on k , the $U(2^i, c_i)$ for $1 \leq i \leq k-1$ are pairwise disjoint, and their union consists of $2^k - 2$ of the 2^k disjoint sets $U(2^k, c)$. By choosing c_k as one of the two remaining values for c , we ensure that the $U(2^i, c_i)$ for $1 \leq i \leq k$ are pairwise disjoint, while not covering all of X . By minimizing $|c_k|$ we ensure that the full collection $\mathcal{C}$ covers X , since $\coprod_{i=1}^{2n+1} U(2^i, c_i)$ is then sure to contain all integers $a \in X$ with $|a| \leq n$.

(i) No, X is not locally compact at any point $a \in X$. We prove this for $a = 0$. If there existed $0 \in V \subset C \subset X$ with V open and C compact, then V would contain a basis element $U(n, 0)$ for some $n \geq 1$, which is closed in X by (b). Since X is Hausdorff by (d), C is closed in X , so $U(n, 0)$ is closed in C and would therefore be compact. However, $f: X \rightarrow U(n, 0)$ given by $f(x) = nx$ is a homeomorphism, and by (h) the space X is not compact, which gives a contradiction.

To see that the bijection f is a homeomorphism, note that it is continuous by (c). Moreover, it is open, because for each basis element $U(m, c) \subset X$ the image $f(U(m, c)) = U(nm, nc)$ is open in X , hence is also open in the subspace $U(n, 0)$.

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