MAT3500/4500 exam 2025 solutions

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Problem 1. (20%) Complete the following definitions:

- **a:** (4%) A topological space X is *compact* if **for each family of open subsets** $(U_\alpha)_{\alpha\in J}$ **with** $\bigcup_{\alpha\in J} U_\alpha = X$ **(a covering) there exist finitely many** $\alpha_1,\dots,\alpha_n \in J$ **with** $U_{\alpha_1}\cup\dots\cup U_{\alpha_n} = X$ **(a finite sub-covering)**. *Variants like "...if each open cover of X contains a finite sub-cover" that do not spell out explicitly what an open cover is are of course also accepted.*
- **b:** (4%) Let X be a topological space and $(x_n)_n \in \mathbb{N}$ a sequence of points $x_n \in X$. The sequence x_n converges to $x \in X$ if **for each neighborhood** U **of** x **there exists** $N \in \mathbb{N}$ **with** $x_n \in U$ **for all** $n \geq N$.
- **c:** (4%) A nonempty topological space X is connected if **there exists no pair of subsets** $U, V \subset X$ **which are open, disjoint, nonempty, and satisfy** $U \cup V = X$**.**
- **d:** (4%) A topological space X is normal if **for each** $x \in X$**,** $\{x\} \subset X$ **is closed, and for each pair** $A, B \subset X$ **of disjoint closed subsets there exist disjoint open subsets** $U, V \subset X$ **with** $A \subset U$ **and** $B \subset V$**.**
- **e:** (4%) Let (X, d) be a metric space. A subset $U \subset X$ is open in the metric topology if **for each point** $x \in U$ **there exists** $\varepsilon > 0$ **such that** $B_d(x, \varepsilon) \subset U$.

Problem 2. (20%) Recall that a subset $T \subset X$ of a topological space is *dense* if $\overline{T} = X$, where $\overline{T}$ denotes the closure of T in X.

Let (X, d) be a metric space, and $T \subset X$ a dense subset. Prove that the collection of subsets

$$\mathcal{B} = \{B_d(x, \tfrac{1}{n}) \mid n \in \mathbb{N}, x \in T\}$$

is a basis for the metric topology on X.

Solution: We need to prove

1. that all $B \in \mathcal{B}$ are open,
2. and that for every open $U \subset X$ and every point $x \in U$, there exists $B \in \mathcal{B}$ with $x \in B \subset U$.

The first statement is clear, since these are balls, and balls are open in the metric topology. To prove the second statement, let U be open and $x \in U$. By definition of the metric topology, there exists $\varepsilon > 0$ with $B_d(x, \varepsilon) \subset U$. Choose n large enough such that $\frac{2}{n} < \varepsilon$. Since $x \in \overline{T}$ and $B_d(x, \frac{1}{n})$ is a neighborhood of x, there exists $t \in T \cap B_d(x, \frac{1}{n})$. So $d(x,t) < \frac{1}{n}$, and hence $x \in B_d(t, \frac{1}{n})$. We also have

$$B_d(t, \frac{1}{n}) \subset B_d(x, \varepsilon) \subset U,$$

since if $y \in B_d(t, \frac{1}{n})$, we have

$$d(y,x) \leq d(y,t) + d(t,x) < \frac{1}{n} + \frac{1}{n} < \varepsilon.$$

So with $B := B_d(t, \frac{1}{n})$ we have $B \in \mathcal{B}$, and $x \in B \subset U$, finishing the proof.

Key steps: Using two suitably nested balls around x (here of radius ε and $\frac{1}{n}$) to find a ball contained in U and an element of t close enough to x, using the triangle inequality to find a ball around t within U containing x. □

Problem 3. (20%) Let X be a topological space and S a set. Call a function $f : X \to S$ *locally constant* if every point $x \in X$ has a neighborhood U such that $f(U)$ consists of a single element. Prove that for a function $f : X \to S$ the following are equivalent:

1. $f : X \to S$ is locally constant.
2. $f : X \to S$ is continuous with respect to the discrete topology on S.

Solution: We first prove $1 \Rightarrow 2$. So let $f : X \to S$ be locally constant, and let $A \subset S$ be an arbitrary subset (since every subset is open in the discrete topology). We need to prove that $f^{-1}(A)$ is open. Let $x \in f^{-1}(A)$. By the definition of locally constant, there exists a neighborhood U_x of x such that $f(U_x)$ has a single element, which must be $f(x)$. Since $x \in f^{-1}(A)$, $f(x) \in A$, and hence $U_x \subset f^{-1}(A)$. Now

$$f^{-1}(A) = \bigcup_{x \in f^{-1}(A)} U_x,$$

which is therefore open as a union of open sets.

Now we prove $2 \Rightarrow 1$. Let $f : X \to S$ be continuous with respect to the discrete topology on S. For any $x \in X$, $\{f(x)\} \subset S$ is open in the discrete topology, and so $U := f^{-1}(\{f(x)\})$ is a neighborhood of x with $f(U) = \{f(x)\}$ consisting of a single element. Therefore f is locally constant.

Key steps: Both directions are relatively directly done from the definitions. For $1 \Rightarrow 2$ we don't directly get that $f^{-1}(A)$ is open, but rather that it contains neighborhoods around every point, so we need the extra argument with the union to conclude that it is open.

If you remembered the notion of "continuous at a point", $1 \Rightarrow 2$ could be done quicker: To check that f is continuous, it suffices to check that it is continuous at every point $x \in X$. To do this, we need to find for every neighborhood V of $f(x)$ a neighborhood U of x with $f(U) \subset V$. But since f is locally constant, we get a neighborhood U with $f(U) = \{f(x)\}$, so this U does the job for every V. □

Problem 4. Let X be a Hausdorff space. Show that

$$\{(x, x) \mid x \in X\} \subset X \times X$$

is a closed subset of $X \times X$ (in the product topology).

Solution: Equivalently, we need to show that the complement

$$W := \{(x, y) \mid x, y \in X, x \neq y\} \subset X \times X$$

is open. For any point $(x, y) \in W$, we have $x \neq y$ by definition of W. By the Hausdorff property, we find $U, V \subset X$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Therefore, $(x, y) \in U \times V \subset W$. Since $U \times V$ is open in the product topology, this shows that any point $(x, y) \subset W$ has a neighborhood contained in W, and so W is open.

Key steps: Translating to the open complement, reducing to finding neighborhoods of each point, invoking the Hausdorff property □

Problem 5. Recall that $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ and $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

Consider the continuous function $f : D^n \to \mathbb{R}^{n+1}$ given by

$$f(x) = (2\sqrt{1 - \|x\|^2}x_1, \ldots, 2\sqrt{1 - \|x\|^2}x_n, 2\|x\|^2 - 1)$$

- **a:** (10%) Show that f corestricts to a continuous function $D^n \to S^n$.
- **b:** (10%) Show that $\widetilde{f}([x]) = f(x)$ defines a homeomorphism $\widetilde{f} : D^n/\sim \to S^n$, where the equivalence relation $\sim$ is defined by

$$x \sim y \text{ if } x = y \text{ or } x, y \in S^{n-1}.$$

Solution: **a:** We only need to check that $f(x) \in S^n$ for each $x \in D^n$, as $S^n \subset \mathbb{R}^{n+1}$ has the subspace topology and a corestriction of a continuous function to a subspace is automatically continuous. We compute

$$\begin{aligned}
\|f(x)\|^2 &= (2\sqrt{1 - \|x\|^2}x_1)^2 + \ldots + (2\sqrt{1 - \|x\|^2}x_n)^2 + (2\|x\|^2 - 1)^2 \\
&= 4(1 - \|x\|^2)(x_1^2 + \ldots + x_n^2) + (4\|x\|^4 - 4\|x\|^2 + 1) \\
&= 4(1 - \|x\|^2)\|x\|^2 + (4\|x\|^4 - 4\|x\|^2 + 1) \\
&= 1,
\end{aligned}$$

so $\|f(x)\| = 1$ and $f(x) \in S^n$ for all x.

Key steps: Mentioning that corestriction to a subspace preserves continuous functions (or doing continuity by hand, using the subspace topology), computing $\|f(x)\| = 1$

b: Consider an element $y = (y_1, \ldots, y_{n+1}) \in S^n$. We compute $f^{-1}(y) \subset D^{n+1}$. First assume $y_{n+1} \neq 1$. If $x \in f^{-1}(y)$, i.e. $f(x) = y$, then

$$2\|x\|^2 - 1 = y_{n+1},$$

so $\|x\| = \sqrt{\frac{y_{n+1}+1}{2}} < 1$. Therefore, $\sqrt{1 - \|x\|^2} > 0$, so $x_1, \ldots, x_n$ are uniquely determined by

$$x_i = \frac{y_i}{2\sqrt{1 - \|x\|^2}}.$$

Conversely, the point $x = (x_1, \ldots, x_n)$ with

$$x_i = \frac{y_i}{2\sqrt{1 - \frac{y_{n+1}+1}{2}}} = \frac{y_i}{\sqrt{2 - 2y_{n+1}}}$$

has

$$\|x\|^2 = \frac{\sum_{i=1}^n y_i^2}{2(1 - y_{n+1})} = \frac{1 - y_{n+1}^2}{2(1 - y_{n+1})} = \frac{y_{n+1} + 1}{2} < 1,$$

and so $x \in D^n$ and $f(x) = y$. This shows that if $y_{n+1} \neq 1$, the preimage $f^{-1}(y)$ consists of a single element.

If $y_{n+1} = 1$, then since $\sum_{i=1}^{n+1} y_i^2 = 1$, $y_1 = \ldots = y_n = 0$. Any preimage $x \in f^{-1}(y)$ has $2\|x\|^2 - 1 = 1$, so $\|x\| = 1$, hence is contained in $S^{n-1} \subset D^n$. Conversely, if $x \in S^{n-1}$, we have $\|x\| = 1$ and so $f(x) = (0, \ldots, 0, 1) = y$. So the preimage of the pole $(0, \ldots, 0, 1)$ is all of S^{n-1}.

Since these preimages are exactly the equivalence classes of $\sim$, we get $f(x) = f(x')$ if and only if $x \sim x'$. So the $\widetilde{f}$ is well-defined and injective. Since the preimage $f^{-1}(y)$ for each $y \in S^n$ is nonempty, $\widetilde{f}$ is also surjective. Since $\pi : D^n \to D^n/\sim$ is a quotient map, $\widetilde{f}$ is continuous if and only if $\widetilde{f} \circ \pi$ is, which it is by **a**. Finally, $D^n/\sim$ is compact (it is the image of the compact D^n unter the continuous function π) and S^n Hausdorff, so a bijective continuous function $\widetilde{f} : D^n/\sim \to S^n$ is automatically a homeomorphism.

Key steps: Well-definedness, bijectivity and continuity of $\widetilde{f}$, the compact-Hausdorff argument.

□