30th October, 2025

MAT3500/4500 — Topology

Solution to Mandatory assignment 1 of 1

Problem 1. Consider the collection $\mathcal{B} = \{[a, b) \subset \mathbb{R} \mid a, b \in \mathbb{R}\}$ of subsets of $\mathbb{R}$.

(a) Prove that $\mathcal{B}$ is the basis of a topology $\mathcal{T}$ on $\mathbb{R}$.

(b) Is $\mathcal{T}$ finer than the metric topology $\mathcal{T}_d$ on $\mathbb{R}$? Is it coarser? Prove or disprove both possible inclusions.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the "floor" function $f(x) = \max\{n \in \mathbb{Z} \mid n \leq x\}$, and $g : \mathbb{R} \to \mathbb{R}$ the "ceiling" function $g(x) = \min\{n \in \mathbb{Z} \mid n \geq x\}$. Considering $\mathbb{R}$ with the topology $\mathcal{T}$ constructed above, which of f or g are continuous?

Solution. (a) We need to check two properties:

(1) For every $x \in \mathbb{R}$, there exists some $B = [a, b) \in \mathcal{B}$ with $x \in [a, b)$. To see this, we can take $[a, b) = [x, x + 1)$.

(2) For $x \in B_1 \cap B_2 = [a, b) \cap [c, d)$, there exists some $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. To see this, first observe that because $x \in [a, b) \cap [c, d)$, we have $a \leq x < b$ and $c \leq x < d$, and $[a, b) \cap [c, d) = [\max(a, c), \min(b, d))$. This is also in $\mathcal{B}$, so we can take $B_3 = [\max(a, c), \min(b, d))$.

(b) We use Lemma 2.2.9 from the lecture notes. First, we claim that $\mathcal{T}_d \subset \mathcal{T}$, so that $\mathcal{T}$ is finer than $\mathcal{T}_d$. Since the open intervals form a basis for the metric topology $\mathcal{T}_d$ on $\mathbb{R}$, we need to show that for each open interval (a, b) and every $x \in (a, b)$, we find some $B \in \mathcal{T}$ with $x \in B \subseteq (a, b)$. We can simply take $[x, b)$. Now, we claim that $\mathcal{T} \not\subset \mathcal{T}_d$, so the $\mathcal{T}_d$ is not finer than $\mathcal{T}$. To see this it suffices to find an open set of $\mathcal{T}$ which is not open in $\mathcal{T}_d$, we claim $[0, 1)$ is such a set. Indeed, this is not open in the metric topology, for example since it contains no ε-ball around 0, since $(-\varepsilon, \varepsilon) \not\subset [0, 1)$ for all $\varepsilon > 0$.

(c) We claim first that f is continuous. To check this, it suffices to check (Lemma 2.6.3 in the lecture notes) that preimages of basis opens are open. Since f only takes integer values, and $f^{-1}(k) = [k, k + 1)$ for $k \in \mathbb{Z}$, we have

$$f^{-1}([a, b)) = \bigcup_{k \in [a,b) \cap \mathbb{Z}} [k, k + 1).$$

The sets $[k, k+1)$ are open in $\mathbb{T}$, so their union as well, and this proves that f is continuous. To see that g is not continuous, we check that $[0, 1)$ is open but $g^{-1}([0, 1)) = (-1, 0]$ is not. $[0, 1)$ is open in $\mathbb{T}$ because

it is one of the basis opens. And $(-1, 0]$ is not open since there is no $[a, b)$ with $0 \in [a, b) \subset (-1, 0]$, as this would mean $0 < b \leq 0$.

Common issues:

- In part (b), sometimes the criterion from Lemma 2.2.9 was applied wrongly, e.g. not for all $x \in (a, b)$ or $x \in [a, b)$ but only for $x \in (x - \varepsilon, x + \varepsilon)$ or $x \in [x - \varepsilon, x + \varepsilon)$.

- Wrong description of the preimages under f and g - these can be described in various ways, but many gave descriptions as an interval where the numbers were "off by one" or only worked for integer a and b.

- When proving a statement of the form "for all $[a, b)$, we have..." like that f is continuous, we have to consider all $[a, b)$. When proving *the negation of such a statement*, like that g is not continuous, meaning "it is not true that for all $[a, b)$, $g^{-1}([a, b))$ is open" we do not have to consider all $[a, b)$, we just have to find one $[a, b)$ for which $g^{-1}([a, b))$ is not open. Not only does this sometimes make things easier, it also removes some potential for mistakes. For example, some people wrote "$g^{-1}([a, b)) = (n, m]$ for $n = \min\{k \in \mathbb{Z} \mid k + 1 \geq a\}$ and $m = \max\{k \in \mathbb{Z} \mid k < b\}$, and $(n, m]$ is not open in $\mathcal{T}$", which is not true in general: For example for $[a, b) = [\frac{1}{2}, 1)$ we get $n = m = 0$ and $(n, m] = \emptyset$, which *is* open. By directly specializing to a specific case of $[a, b)$ which suffices we avoid such mistakes and make the argument easier to read as well.

Problem 2. Let X be a topological space, and $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ continuous functions.

(a) Prove that $\{x \in X \mid f(x) = 0\}$ is a closed subset of X.

(b) Prove that $\{x \in X \mid f(x) > 0\}$ is an open subset of X.

(c) Prove that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

(d) Prove that $\{x \in X \mid f(x) \leq g(x)\}$ is a closed subset of X.

(e) Prove that $\{(x, y) \in X \times X \mid f(x) = g(y)\}$ is a closed subset of $X \times X$.

Solution. (a) $\{0\}$ is a closed subset of $\mathbb{R}$. Since preimages of closed subsets under continuous functions are closed,

$$\{x \in X \mid f(x) = 0\} = f^{-1}(\{0\})$$

is also closed.

(b) Similarly, this is $f^{-1}((0, \infty))$ and therefore open.

(c) This set agrees with $h^{-1}(\{0\})$ for $h(x) = f(x) - g(x)$. This h is the difference of two continuous functions and therefore also continuous, so $h^{-1}(\{0\})$ is closed. (To see that h is continuous, we can write it as the composition of the continuous functions $x \mapsto (f(x), g(x)), X \to \mathbb{R}^2$ and the function $(x, y) \mapsto x - y, \mathbb{R}^2 \to \mathbb{R}$.)

(d) For the same h as in the previous exercise, we can write this set as $h^{-1}((-\infty, 0])$, so it is closed.

(e) Here, we construct a function $h : X \times X \to \mathbb{R}$ with $h(x, y) = f(x) - g(y)$. It is continuous because we can write this as difference of the continuous functions $f \circ \pi_1 : X \times X \to \mathbb{R}$ and $g \circ \pi_2 : X \times X \to \mathbb{R}$, with $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ the projections, which are continuous as we have proved in the chapter about the product topology.

Common issues:

None really!

Problem 3. Let X and Y be topological spaces, $x_n \in X$ and $y_n \in Y$ sequences of points, and $x \in X$, $y \in Y$ additional points. Prove that the following two statements are equivalent:

1. The sequence x_n converges to x in X, and the sequence y_n converges to y in Y.

2. The sequence (x_n, y_n) converges to (x, y) in $X \times Y$ (with the product topology).

Solution. We prove both implications separately:

1.⇒2. Assuming that x_n converges to x and y_n converges to y, we need to check that (x_n, y_n) converges to (x, y) the following statement: *For all neighborhoods W of (x, y) in $X \times Y$, there exists N such that $(x_n, y_n) \in W$ for all $n \geq N$.* So let W be a neighborhood of (x, y) in $X \times Y$. Since a basis of the product topology is given by products $U \times V$ with open U and V, we find such a neighborhood of (x, y), meaning we find $U \subset X$ and $V \subset Y$ open with $(x, y) \in U \times V \subset W$. Now since x_n converges to x, we find N_1 such that $x_n \in U$ for all $n \geq N_1$, and since y_n converges to y, we find N_2 such that $y_n \in V$ for all $n \geq N_2$. Take $N = \max(N_1, N_2)$, then $(x_n, y_n) \in U \times V \subset W$ for all $n \geq N$ as claimed.

2.⇒1. Assuming that (x_n, y_n) converges to (x, y), we prove that x_n converges to x (the proof that y_n converges to y is completely analogous). We need to prove the following statement: *For all neighborhoods U of x in X, there exists N such that $x_n \in U$ for all $n \geq N$.* So let U be an arbitrary neighborhood of x in X. Then $U \times Y$ is a neighborhood of (x, y) in $X \times Y$. So by assumption we find N with $(x_n, y_n) \in U \times Y$ for each $n \geq N$, so $x_n \in U$ for all $n \geq N$. This proves that x_n converges to x.

Remark: For 2.⇒1., some made the clever observation that this follows directly from continuity of the projections and the fact that continuous functions preserve limits, since $\pi_1(x_n, y_n) = x_n$ and $\pi_1(x, y) = x$.

Common issues:

- A lot of people missed the step in 1.⇒2. that convergence in $X \times Y$ is a statement about all neighborhoods of (x, y), but not all of those are of the form $U \times V$, so one needs to use that these are a basis to first reduce to that case.

- There was some confusion about the definition of convergence/limits. For example, for $x_n \to x$, it is not enough to know that every open neighborhood of x contains infinitely many x_n. Also, the order of quantifiers really matters: The correct statement is of the form "For all U, there exists N,...", which means that the N can be a different one for each choice of U, so N can depend on U. The analogous statement of the form "There exists N such that for all U ..." is completely different and would actually imply that the sequence becomes constant at some point if we are in a Hausdorff space (that is a fun short exercise).

Problem 4. For each of the following pairs of topological spaces, find an explicit homeomorphism between them.

(a) $\mathbb{R} - \{0\}$ and $(1,2) \cup (3,4)$

(b) $\mathbb{R}^2 - \{0\}$ and $\mathbb{R} \times S^1$

Solution. (a) There are many different homeomorphisms one can choose here, we give one example: Let $f : \mathbb{R} - \{0\} \to (1,2) \cup (3,4)$ be given by

$$f(x) = \begin{cases} 1 + e^{-x} \text{ for } x > 0 \\ 3 + e^{x} \text{ for } x < 0 \end{cases}$$

Then since $0 < e^{-x} < 1$ for $x > 0$ and $0 < e^{x} < 1$ for $x < 0$, f takes values in $(1,2) \cup (3,4)$. On $(0,\infty)$, f is given by $1 + e^{-x}$, which is continuous since it is built from standard continuous functions using composition, sums etc. One $(-\infty, 0)$, it is given by $3 + e^{x}$ which is also continuous by the same type of argument. Since $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$, f is continuous. (Formally, we can use Theorem 2.6.27 from the notes here, which allows us to check continuity on each of the opens covering a space individually, here $(0,\infty)$ and $(-\infty, 0)$.) Now let $g : (1,2) \cup (3,4) \to \mathbb{R} - \{0\}$ be given by

$$g(x) = \begin{cases} -\ln(x-1) \text{ for } x \in (1,2) \\ \ln(x-3) \text{ for } x \in (3,4) \end{cases}$$

Since ln takes negative values on $(0,1)$, g indeed takes values in $\mathbb{R} - \{0\}$. It is also again continuous since we directly see that its restriction to the opens $(1,2)$ and $(3,4)$ is continuous. Finally, one directly checks $f(g(x)) = x$ and $g(f(x)) = x$, so both f and g are bijective and inverse to each other. It follows that f is a homeomorphism and $g = f^{-1}$.

(b) We again define our homeomorphism and a map in the other direction and check that both of them are continuous and inverse to each other. Let $f : \mathbb{R}^2 - \{0\} \to \mathbb{R} \times S^1$ be given by

$$f(x,y) = \left(\ln(\|(x,y)\|), \frac{x}{\|(x,y)\|}, \frac{y}{\|(x,y)\|}) \right)$$

with $\|(x,y)\| = \sqrt{x^2 + y^2}$. This takes values in $\mathbb{R} \times S^1$ since

$$\left\| \left(\frac{x}{\|(x,y)\|}, \frac{y}{\|(x,y)\|} \right) \right\| = \frac{\|(x,y)\|}{\|(x,y)\|} = 1.$$

It is also continuous, since $\ln(\|(x,y)\|)$ etc. are continuous. (Formally we use here that a function to the subspace $\mathbb{R} \times S^1 \subset \mathbb{R}^3$ is continuous if and only if it is continuous when viewed as function to $\mathbb{R}^3$, and using the universal property of the product topology we may check continuity of a function to $\mathbb{R}^3$ by checking continuity of the three functions to $\mathbb{R}$ we get by composing with the projections.)

In the other direction, we define a function $g : \mathbb{R} \times S^1 \to \mathbb{R}^2 - \{0\}$, via

$$g(s,x,y) = (e^s x, e^s y),$$

which takes values in $\mathbb{R}^2 - \{0\}$ since if $(e^s x, e^s y) = 0$, $e^s \neq 0$ tells us $(x,y) = (0,0)$, but $(0,0) \notin S^1$. This g is again continuous, because it is obtained by corestriction and restriction from the function $\mathbb{R}^3 \to \mathbb{R}^2$ given by the same formula, whose continuity follows again because it is built from standard functions using products etc.

Now $g(f(x,y)) = (x,y)$ is easy to see. For $f \circ g$, we observe that if $(x,y) \in S^1$, then $\|(e^s x, e^s y)\| = e^s$, and so

$$f(g(s,x,y)) = \left(\ln(e^s), \frac{e^s x}{e^s}, \frac{e^s y}{e^s}\right) = (s,x,y).$$

So f and g are bijective and inverse to each other, and so f is a homeomorphism.

Common issues:

- Often, formulas for a homeomorphism were given without justification that they actually provide a homeomorphism. "This is clearly continuous by calculus" is fine to say as long as the function is given by as a function $f : X \to Y$ where $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, and each of the m functions $f_1, \ldots, f_m : X \to \mathbb{R}$ making up f are given by some sort of elementary formula, meaning it is built from the standard functions from calculus using composition and operations like addition, subtraction, multiplication and division. However, already when the functions are specified by a piecewise description, it can become difficult to decide continuity using the methods of calculus. In those cases, you probably want to use criteria like Theorems 2.6.27 and 2.6.28 from the lecture notes, which allow you to check continuity on each of the pieces, *but keep in mind there are actually some conditions to check!*
- The homeomorphism $\mathbb{R}^2 - \{0\} \cong \mathbb{R} \times S^1$ is closely related to polar coordinates, *but you should not try to produce it using polar coordinates!*

Several people tried to argue that polar coordinates give you some bijection between $\mathbb{R}^2 - \{0\}$ and $\mathbb{R}_{>0} \times [0, 2\pi)$, taking (x, y) to the corresponding polar coordinates (r, θ), and also a bijection $S^1 \cong [0, 2\pi)$, and that the homeomorphism and its inverse can be described with respect to these bijections as

$$(r, \theta) \mapsto (\ln(r), \theta)$$

and

$$(s, \theta) \mapsto (e^s, \theta).$$

While that is actually what the homeomorphism from the above discussion looks like in polar coordinates, it is useless: Both the bijection $\mathbb{R}^2 - \{0\} \to \mathbb{R}_{>0} \times [0, 2\pi)$ and $S^1 \to [0, 2\pi)$ are not continuous, soknowing that the above functions between $\mathbb{R}_{>0} \times [0, 2\pi)$ and $\mathbb{R} \times [0, 2\pi)$ are continuous tells you nothing about the corresponding functions between $\mathbb{R}^2 - \{0\}$ and $\mathbb{R} \times S^1$. If you however directly try to write down this function in terms of $S^1 = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\}$, these discontinuities go away.

(This is why S^1 is better than $[0, 2\pi)$ for polar coordinates, and in general it is a better idea to encode directions as vectors in S^1 instead of angles in $[0, 2\pi)$, for example for applications in physics, engineering, computer graphics, machine learning,...)

Problem 5. Let X be a set, and d, d' two metrics on X. Assume there exist constants $c, C \in \mathbb{R}_{>0}$ such that

$$c \cdot d'(x,y) < d(x,y) < C \cdot d'(x,y)$$

for all $x, y \in X$. Show that then the metric topologies $\mathcal{T}_d, \mathcal{T}_{d'}$ induced by the two metrics on X agree.

Solution. Because $d(x,y) < Cd'(x,y)$, we have that if $d'(x,y) < \frac{\varepsilon}{C}$, then $d(x,y) < \varepsilon$. This means that

$$B_{d'}(x, \tfrac{\varepsilon}{C}) \subset B_d(x, \varepsilon).$$

for any x and ε. Similarly,

$$B_d(x, c\varepsilon) \subset B_{d'}(x, \varepsilon).$$

Now we directly check that $\mathcal{T}_d = \mathcal{T}_{d'}$. Recall that a set U is open in the metric topology $\mathcal{T}_d$ if and only if, for every $x \in U$, we find an $\varepsilon > 0$ with $B_d(x, \varepsilon) \subset U$. But then also $B_{d'}(x, \varepsilon/C) \subset U$, so we also find a d'-ball around each x, and therefore U is also open in $\mathcal{T}_{d'}$. The other direction works analogously using the other inclusion of balls.

Common issues:

- A lot of people proved the above inclusion relation between d-balls and d'-balls, but just not explained why that proves that the topologies agree.
- Some tried to use Lemma 2.2.9 from the lecture notes (or Corollary 2.2.10). That requires you to know that for every basis element B in one topology and *every* $x \in B$, we find a B' in the other basis with $x \in B' \subset B$. Translated to the present situation, we would need to know that for every d-ball $B_d(x, \varepsilon)$ and *every* point $y \in B_d(x, \varepsilon)$, we find a d'-ball with $y \in B_{d'}(z, \delta) \subset B_d(x, \varepsilon)$. The inclusion in the solution above only provides this for the case $y = x$. One can also use Lemma 2.2.9 here, but it requires an additional application of the triangle inequality. It is much easier to directly use the definition of open sets in the metric topology (Definition 2.1.15) directly, then we get away with just comparing d-balls and d'-balls with the same center x.

Problem 6. Let $X = \mathbb{R} \times \{0, 1\}$ with the product topology, where $\{0, 1\}$ carries the discrete topology, and consider the equivalence relation $\sim$ on X defined by

$$(x, i) \sim (y, j) \text{ if } x = y \text{ and } x \neq 0, \text{ or } x = y \text{ and } i = j.$$

Prove that $X/\sim$ does not have the Hausdorff property.

Solution. Writing $\pi : X \to X/\sim$ for the canonical projection, we have that $\pi(0, 0) \neq \pi(0, 1)$, since $(0, 0)$ and $(0, 1)$ are not equivalent under $\sim$. We will prove that for any pair of open neighborhoods $U \ni \pi(0, 0)$, $V \ni \pi(0, 1)$, we have $U \cap V \neq \emptyset$, thereby proving that $X/\sim$ is not Hausdorff. So let U and V be such neighborhoods. We let $\widetilde{U} = \pi^{-1}(U)$ and $\widetilde{V} = \pi^{-1}(V)$. These are open in $X = \mathbb{R} \times \{0, 1\}$ and contain $(0, 0)$ and $(0, 1)$ respectively. By the definition of the product topology, $\widetilde{U}$ contains a neighborhood of $(0, 0)$ of the form $U_1 \times U_2$ with U_1 a neighborhood of 0 in $\mathbb{R}$, and U_2 a subset of $\{0, 1\}$ containing 0. Since U_1 is open in $\mathbb{R}$ means that we find $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times \{0\} \subset U_1 \times U_2 \subset \widetilde{U}$. Analogously, $\widetilde{V}$ contains $(-\delta, \delta) \times \{1\}$ for some $\delta > 0$. For $t = \frac{\min(\varepsilon, \delta)}{2}$, we have therefore that $(t, 0) \in \widetilde{U}$ and $(t, 1) \in \widetilde{V}$, so $\pi(t, 0) \in U$ and $\pi(t, 1) \in V$. But since $t \neq 0$, $(t, 0) \sim (t, 1)$, so $\pi(t, 0) = \pi(t, 1)$ and we have found a common element in U and V, proving $U \cap V \neq \emptyset$.

And here is a very slick alternative solution that I did not think about but saw in solutions from two different people:

Solution. In $\mathbb{R}$, the sequence $\frac{1}{n}$ converges to 0. So in $X = \mathbb{R} \times \{0, 1\}$, the sequence $(\frac{1}{n}, 0)$ converges to $(0, 0)$ and the sequence $(\frac{1}{n}, 1)$ converges to $(0, 1)$ (for example by Problem 3). Now the canonical projection $\pi : X \to X/\sim$ is continuous because it is a quotient map by construction, and so it preserves limits of sequences. So $\pi(\frac{1}{n}, 0)$ converges to $\pi(0, 0)$, and $\pi(\frac{1}{n}, 1)$ converges to $\pi(0, 1)$. But since $(\frac{1}{n}, 0) \sim (\frac{1}{n}, 1)$, $\pi(\frac{1}{n}, 0) = \pi(\frac{1}{n}, 1)$, so both sequences are the same. However, $(0, 0)$ and $(0, 1)$ are not equivalence, and so $\pi(0, 0) \neq \pi(0, 1)$ and the sequence has two different limits. By uniqueness of limits in Hausdorff spaces (Theorem 2.5.32 in the notes), this means that $X/\sim$ cannot be Hausdorff.

Common issues:

- Some people were not clear enough about when they were talking about points and subsets of $X/\sim$, and when about points and subsets of X. It helps to denote the image of some point $x \in X$ in $X/\sim$ either as $\pi(x)$ (as in the solution above, using the canonical projection), or as $[x]$ (using equivalence class notation).

- Some people were confused about the correct negation of the Hausdorff property: Since the Hausdorff property says "For all $x \neq y$, there exist neighborhoods U, V, such that $U \cap V = \emptyset$", its negation (meaning a space *not* being Hausdorff) says "There exist $x \neq y$ such that for all neighborhoods U and V, $U \cap V \neq \emptyset$". So we have to find two specific points, but then we have to show that *all* pairs of neighborhoods of those points are not disjoint.

Problem 7. Find an example of a quotient map $f : X \to Y$ which is neither open nor closed.

Solution. We consider $Y = \{a, b\}$ with topology $\mathcal{T} = \{\emptyset, \{a\}, Y\}$ the Sierpinski topology, $X = \mathbb{R}$ with the usual topology, and $f : X \to Y$ given by

$$f(x) = \begin{cases} a \text{ if } x < 0 \\ b \text{ if } x \geq 0. \end{cases}$$

Then f is a quotient map: $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(Y) = X$, $f^{-1}(\{a\}) = (-\infty, 0)$, $f^{-1}(\{b\}) = [0, \infty)$ of which all except the last one are open, so we see that a subset of Y is open if and only if its preimage under f is open (just by looking at all four subsets explicitly). Now f is not open, since $(0, 1)$ is open in $\mathbb{R}$ but its image $f((0, 1)) = \{b\}$ is not open in Y, and f is not closed since $[-2, -1]$ is closed in $\mathbb{R}$ but its image $f([-2, -1]) = \{a\}$ is not closed in Y.

Remark: There were a lot of great examples here: A lot of them were various quotients of $\mathbb{R}$, but some people also handcrafted weird topologies on a 4-element set for X and a 2-element set for Y, and someone composed an open map with a closed map and analyzed the result of that. Surprisingly, very few people did the example from Exercise 22.3. Do the exercises!!

Common issues:

- When analyzing that $f : X \to Y$ is neither open nor closed, we have to find an open set U such that $f(U)$ is *not open* and a closed set A such that $f(A)$ is *not closed*. A lot of solutions instead produced an open U such that $f(U)$ is *closed* and a closed A such that $f(A)$ is *open*. This is probably the (understandably very tempting) misconception that "open" and "closed" are opposites, but of course a subset can also be neither or both of these things. (For example $\emptyset$ and the entire space are always open and closed, and we have seen in the discussion of connectedness that in a non-connected space there are more subsets which are simultaneously open and closed.)