5th October, 2025

MAT3500/4500 — Topology

Mandatory assignment 1 of 1

Submission deadline

Thursday 23rd October 2025, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

For MAT4500 students, the mandatory assignment must be written with a typesetting software for writing mathematics (e.g. LaTeX). MAT3500 students can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LATEX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

The assignment consists of 7 parts, each worth 10 points. Each answer must be justified. A total score of 40 %, or 28 points, is sufficient to pass.

Problem 1. Consider the collection $\mathcal{B} = \{[a, b) \subset \mathbb{R} \mid a, b \in \mathbb{R}\}$ of subsets of $\mathbb{R}$.

(a) Prove that $\mathcal{B}$ is the basis of a topology $\mathcal{T}$ on $\mathbb{R}$.

(b) Is $\mathcal{T}$ finer than the metric topology $\mathcal{T}_d$ on $\mathbb{R}$? Is it coarser? Prove or disprove both possible inclusions.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the "floor" function $f(x) = \max\{n \in \mathbb{Z} \mid n \leq x\}$, and $g : \mathbb{R} \to \mathbb{R}$ the "ceiling" function $g(x) = \min\{n \in \mathbb{Z} \mid n \geq x\}$. Considering $\mathbb{R}$ with the topology $\mathcal{T}$ constructed above, which of f or g are continuous?

Problem 2. Let X be a topological space, and $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ continuous functions.

(a) Prove that $\{x \in X \mid f(x) = 0\}$ is a closed subset of X.

(b) Prove that $\{x \in X \mid f(x) > 0\}$ is an open subset of X.

(c) Prove that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

(d) Prove that $\{x \in X \mid f(x) \leq g(x)\}$ is a closed subset of X.

(e) Prove that $\{(x, y) \in X \times X \mid f(x) = g(y)\}$ is a closed subset of $X \times X$.

Problem 3. Let X and Y be topological spaces, $x_n \in X$ and $y_n \in Y$ sequences of points, and $x \in X$, $y \in Y$ additional points. Prove that the following two statements are equivalent:

1. The sequence x_n converges to x in X, and the sequence y_n converges to y in Y.
2. The sequence (x_n, y_n) converges to (x, y) in $X \times Y$ (with the product topology).

Problem 4. For each of the following pairs of topological spaces, find an explicit homeomorphism between them.

(a) $\mathbb{R} - \{0\}$ and $(1, 2) \cup (3, 4)$

(b) $\mathbb{R}^2 - \{0\}$ and $\mathbb{R} \times S^1$

Problem 5. Let X be a set, and d, d' two metrics on X. Assume there exist constants $c, C \in \mathbb{R}_{>0}$ such that

$$c \cdot d'(x, y) < d(x, y) < C \cdot d'(x, y)$$

for all $x, y \in X$. Show that then the metric topologies $\mathcal{T}_d, \mathcal{T}_{d'}$ induced by the two metrics on X agree.

Problem 6. Let $X = \mathbb{R} \times \{0, 1\}$ with the product topology, where $\{0, 1\}$ carries the discrete topology, and consider the equivalence relation $\sim$ on X defined by

$$(x, i) \sim (y, j) \text{ if } x = y \text{ and } x \neq 0, \text{ or } x = y \text{ and } i = j.$$

Prove that $X/\sim$ does not have the Hausdorff property.

Problem 7. Find an example of a quotient map $f : X \to Y$ which is neither open nor closed.