

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4500 — Topology

Day of examination: Thursday December 17, 2015

Examination hours: 09:00am – 01:00pm

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

We claim that 0 and $\sqrt{2}$ are the only limit points of $A = (0, \sqrt{2})$.

Every basic neighborhood of 0 has the form $[a, b)$, where $a \leq 0 < b$ are rational numbers. Then $b/2 \in [a, b) \cap A$, so it follows that 0 is a limit point. Since $\sqrt{2}$ is an irrational number, every basic neighborhood of $\sqrt{2}$ has the form $[a, b)$, where $a < \sqrt{2} < b$ are rational numbers. Now every real number r between a and $\sqrt{2}$ lies in the intersection $[a, b) \cap A$, so it follows that $\sqrt{2}$ is a limit point.

If $r > \sqrt{2}$ there exists a rational number s such that $\sqrt{2} < s < r$. Note that $[s, r + 1)$ is a neighborhood of r disjoint from A . If $r < 0$, then $[r, 0)$ is a neighborhood of r disjoint from A .

b

We claim that $\sqrt{2}$ is the only limit point of $B = (\sqrt{2}, 3)$.

Every basic neighborhood of $\sqrt{2}$ has the form $[a, b)$, where $a < \sqrt{2} < b$ are rational numbers. Choose a rational number $r < 3$ between $\sqrt{2}$ and b . Then $r \in [a, b) \cap B$, so it follows that $\sqrt{2}$ is a limit point. If $r \geq 3$, then $[r, r + 1)$ is a neighborhood of r disjoint from B . If $r < \sqrt{2}$, there exists rational numbers s, t such that $s < r < t < \sqrt{2}$. Note that $[s, t)$ is a neighborhood of r disjoint from B .

Remark More generally, by the same reasoning, we have

$$\overline{(a, b)} = \begin{cases} [a, b) & \text{if } b \in \mathbf{Q}, \\ [a, b] & \text{if } b \notin \mathbf{Q}. \end{cases}$$

See also §17, Exercise 17.

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Problem 2

a

See §23.

b

Suppose $[0, 1] = U \cup V$, where U and V are nonempty disjoint open subsets. We may assume $0 \in U$. Let s be the supremum of $\{t \mid [0, t] \in U\}$. If $s \in U$, there exists an $\epsilon > 0$ such that the set of all $x \in [0, 1]$ with $|x - s| < \epsilon$ is contained in U . This contradicts the definition of s in case $s < 1$, because then there exists an $s' > s$ such that $[0, s'] \in U$. However, if $s = 1$, then $[0, 1] \subset U$, contradicting the choices of U and V . We conclude that $s \in V$. In this case, we choose $\epsilon > 0$ such that $x \in [0, 1]$ and $|x - s| < \epsilon$ implies $x \in V$. By definition of s we can find s' with $[0, s'] \in U$ and $|s' - s| < \epsilon$. This contradiction shows that U and V as above cannot exist.

Problem 3

a

See the proof of Theorem 60.3.

b

Note that $g: X \rightarrow f(X)$ is a continuous bijection. We want to show that g^{-1} is continuous. Any closed subset $C \subset X$ is compact because X is compact. By continuity of g it follows that $g(C)$ is compact. Note that $f(X) \subset Y$ is Hausdorff (subspace of a Hausdorff space). Thus $g(C)$ is closed in $f(X)$. This shows that g^{-1} is continuous.

c

The map f factors through $\mathbf{RP}^2$ since $f(-\mathbf{x}) = f(\mathbf{x})$. It suffices to show the induced map $\bar{f}: \mathbf{RP}^2 \rightarrow \mathbf{R}^4$ is injective. In complex coordinates, the self-map of $\mathbf{R}^2$ given by $(x, y) \mapsto (x^2 - y^2, 2xy)$ is just $z \mapsto z^2$. Note that $z_1^2 = z_2^2$ if and only if $z_1 = \pm z_2$. Suppose (x, y, z) and (x', y', z') on S^2 maps to the same point in $\mathbf{R}^4$ under f . Then $(x, y) = \pm(x', y')$ by the above. If $x = x' \neq 0$, the equation $xz = x'z'$ implies $z = z'$. If $x = -x' \neq 0$, the equation $xz = x'z'$ implies $z = -z'$. Similar arguments apply if $y = \pm y' \neq 0$ because of the equation $yz = y'z'$. If $x = y = x' = y' = 0$ then $z = \pm 1$ and $z' = \pm 1$, and thus $z = \pm z'$.

d

Note that every nonzero $\mathbf{x}$ is equivalent to $\mathbf{x}/\|\mathbf{x}\| \in S^{n-1}$. Thus the set of equivalence classes identifies with the points on the sphere $S^{n-1} \subset \mathbf{R}^n$ and the one-point space $\{0\} \subset \mathbf{R}^n$.

If U is a neighborhood of 0, then $\pi^{-1}(U) \subset \mathbf{R}^n$ is open and hence contains at least one point equivalent to any given point in $\mathbf{R}^n$. Here, $\pi: \mathbf{R}^n \rightarrow Y$ is

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the quotient map. It follows that $U = Y$. Thus 0 cannot be separated from any other element of Y , i.e., Y is not Hausdorff.

Problem 4

a

Define $H: \mathbf{M} \times I \rightarrow \mathbf{M}$ by $H(s, t, r) = (s, (1-r)t)$. Note that $(0, t)$ is mapped to $(0, (1-r)t)$ and $(1, -t)$ is mapped to $(1, -(1-r)t)$, i.e., the homotopy is well-defined. At time $r = 0$, H is the identity map. At time $r = 1$, H sends $\mathbf{M}$ to $S_{\mathbf{M}}^1$. Finally, H is the identity on $S_{\mathbf{M}}^1$ since $H(s, 0, r) = (s, 0)$.

b

From **a**) it follows that $\pi_1 \mathbf{M}$ is infinite cyclic.

THE END