

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

New exam in: MAT3500/4500 — Topology

Day of examination: Tuesday December 6, 2022

Examination hours: 15.00–19.00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Convention:

- $\mathbb{R}$: the set of real numbers
- I : closed unit interval $[0, 1]$
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{C}$: the set of complex numbers
- S^1 : unit circle in the Euclidean plane; $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$
- B^2 : the closed unit disc in the Euclidean plane;
 $B^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$

We consider them as topological spaces in the conventional topology unless stated otherwise. (On $\mathbb{Q}$, as you can see below, we consider the induced topology for the inclusion $\mathbb{Q} \subset \mathbb{R}$.)

Part I

Answer all questions from this part. These are all multiple-choice questions, and you can just state your answer without justification.

Problem 1 (weight 5 points)

Consider the collection of subsets of $X = [-1, 1]$:

$$\mathcal{T} = \{X \cap [a, b) \mid a, b \in \mathbb{R}\}.$$

This cannot be interpreted as the collection of open sets for some topology on X . Choose an appropriate reason.

1. $(0, 1)$ is the union of $[1/n, 1)$, so $\mathcal{T}$ is not closed under union.
2. $[0, 1]$ is the complement of $[-1, 0)$ in X , so $\mathcal{T}$ is not closed under complement.
3. None of the above

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Problem 2 (weight 5 points)

Which of the following is closed in $\mathbb{R}$?

1. $A = \bigcap_{n=1}^{\infty} A_n$, where

$$A_n = [0, 1/3^n] \cup [2/3^n, 3/3^n] \cup [2/3^{n-1}, 7/3^n] \cup \dots \cup [(3^n - 1)/3^n, 1];$$

A_n is the union of 2^n intervals of length $1/3^n$, and A is the intersection of the A_n .

2. $B = \mathbb{Q}$;
3. $C = \bigcup_{n=1}^{\infty} [0, 1 - 1/n]$; union of closed intervals of length $1 - 1/n$.

Problem 3 (weight 5 points)

Which of the following is a base of the usual topology on $\mathbb{R}$?

1. $\mathcal{T} = \{[a, b] \mid a, b \in \mathbb{R}\}$
2. $\mathcal{T}' = \{A \subset \mathbb{R}\}$ (collection of all subsets)
3. None of the above

Problem 4 (weight 5 points)

Let (X, d) be a metric space. Suppose that points x, y, z in X satisfy $d(x, y) = 1$ and $d(y, z) = 2$. What can we say about $d(x, z)$?

1. $1 \leq d(x, z) \leq 3$
2. $d(x, z) \geq 3$
3. $d(x, z) = 1$

Problem 5 (weight 5 points)

Let X and Y be topological spaces. What can we say about the product topology on $X \times Y$?

1. Open sets are unions of the sets of the form $U \times V$, where U is open in X and V is open in Y .
2. Open sets are of the form $U \times V$, where U is open in X and V is open in Y .
3. None of the above

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Problem 6 (weight 5 points)

The real function

$$f(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0) \end{cases}$$

is not continuous. What is the appropriate reason for this?

1. Consider the open set $U = (1/2, 3/2)$ of $\mathbb{R}$. The inverse image $f^{-1}(U)$ is not open in $\mathbb{R}$.
2. Consider the open set $V = (-1/2, 1/2)$ of $\mathbb{R}$. The inverse image $f^{-1}(V)$ is not open in $\mathbb{R}$.
3. None of the above

Problem 7 (weight 5 points)

Consider the following sequence of real functions f_n :

$$f_n(x) = \begin{cases} n|x| & (|x| \leq 1/n) \\ 1 & (|x| > 1/n) \end{cases}$$

What can we say about this?

1. This converges to $f(x) = 0$ or 1 according to $x = 0$ or not, in the topology of pointwise convergence.
2. This converges to $f(x) = 0$ or 1 according to $x = 0$ or not, in the topology of uniform convergence.
3. This converges to the constant function $f(x) = 1$ in any sensible topology.

Problem 8 (weight 5 points)

Consider the following subsets of $\mathbb{R}^2$. Which one is connected for the induced topology?

1. $A = \{(x, y) \mid x^2 + y^2 \leq 1 \text{ or } (x - 2)^2 + y^2 \leq 1\}$
2. $B = \{(x, 0) \mid x \in \mathbb{Q}\}$
3. $C = \{(x, \sin x) \mid x \in \mathbb{R}\}$

Problem 9 (weight 5 points)

What can we say about the infinite product space $X = \prod_{i=1}^{\infty} \{0, 1\} = \{(x_i)_{i=1}^{\infty} \mid x_i = 0 \text{ or } 1\}$?

1. This space is compact, because it is the product of (copies of) the compact space $\{0, 1\}$.
2. This space is not compact, because it is not a closed bounded subset of $\mathbb{R}^k$.
3. None of the above

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Problem 10 (weight 5 points)

Which one is correct?

1. The loop in S^1 given by $f: I \rightarrow S^1, t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is not path-homotopic to the loop $g: I \rightarrow S^1, t \mapsto (\cos 4\pi t, \sin 4\pi t)$.
2. The loop in B^2 given by $f: I \rightarrow B^2, t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is not path-homotopic to the loop $g: I \rightarrow B^2, t \mapsto (\cos 4\pi t, \sin 4\pi t)$.
3. None of the above

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Part II

In this part, justify your answer with appropriate amount of reasoning. Choose two out of the four problems from this part. You may choose more than two to answer, but we only use the best two ones among your answers to decide the score.

Problem 11 (weight 25 points)

Let $X = \prod_J \mathbb{R}$ be the direct product of uncountably many copies of $\mathbb{R}$, with the product topology. Thus, the index set J is an uncountable set. As a subspace of X , we consider

$$A = \{(x_i)_{i \in J} \mid x_i = 1 \text{ except for finitely many } i \in J\}.$$

1. Let x^0 denote the constant 0 sequence in X . (That is, $x^0 = (x_i^0)_i \in X$ defined by $x_i^0 = 0$ for all $i \in J$.) Show that x^0 belongs to the closure of A .
2. Let $(a^n)_{n=1}^\infty$ be a sequence in A , and for each n put

$$J_n = \{i \in J \mid a_i^n \neq 1\}.$$

Show that there is an element $\beta \in J$ not in $\bigcup_n J_n$.

3. Show that $(a^n)_n$ does not converge to x^0 .

Problem 12 (weight 25 points)

We say that a continuous map $f: X \rightarrow Y$ is *proper* if, for any compact subset $A \subset Y$, its inverse image $f^{-1}(A)$ is proper.

1. Give an example of a continuous map that is not proper.

Now, suppose that X and Y are locally compact spaces. Recall that we have one-point compactifications $X^+ = X \cup \{\infty\}$, $Y^+ = Y \cup \{\infty\}$ by formally adding extra points, and setting $(X \setminus K) \cup \{\infty\}$ for compact $K \subset X$ as open neighborhoods of ∞ in X^+ , etc.

2. Given a continuous map $f: X \rightarrow Y$, consider the map $f^+: X^+ \rightarrow Y^+$ defined by

$$f^+(x) = \begin{cases} f(x) & (x \in X) \\ \infty & (x = \infty). \end{cases}$$

Show that f^+ is continuous when f is proper.

3. Find a (non-proper) continuous map f between locally compact spaces such that the above f^+ is not continuous.

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Problem 13 (weight 25 points)

We want to prove the following variation of the fundamental theorem of algebra: let $p(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients of degree $n > 0$, such that $|a_{n-1}| + \cdots + |a_0| < 1$. Then there is a complex number z such that $|z| < 1$ and $p(z) = 0$.

Identify S^1 with the subset $\{z \in \mathbb{C} \mid |z| = 1\}$ of $\mathbb{C}$, and similarly B^2 with $\{z \in \mathbb{C} \mid |z| \leq 1\}$. We also put $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

1. Assume that there is no z such that $|z| < 1$ and $p(z) = 0$. Show that the map $h: S^1 \rightarrow \mathbb{C}^\times, z \mapsto p(z)$ is homotopic to a constant map.
2. Under the same assumption, show that h is homotopic to the map $k: S^1 \rightarrow \mathbb{C}^\times, z \mapsto z^n$.
3. Derive a contradiction from the above.

Problem 14 (weight 25 points)

Given a commutative ring A , recall that we obtained a topological space

$$\operatorname{Spec} A = \{P \subset A \mid P \text{ is a prime ideal of } A\}$$

whose closed sets are given by

$$V(I) = \{P \in \operatorname{Spec} A \mid P \supset I\} \subset \operatorname{Spec} A.$$

1. Let $f: A \rightarrow B$ be a homomorphism of commutative rings. Show that the map $f^*: \operatorname{Spec} B \rightarrow \operatorname{Spec} A, P \mapsto f^{-1}(P)$ is continuous.
2. Given $P \in \operatorname{Spec} A$, let K be the field of fractions of the integral domain $O = A/P$. What is the continuous map $\operatorname{Spec} K \rightarrow \operatorname{Spec} A$ corresponding to the natural homomorphism $A \rightarrow K$?
3. Let A be a Noetherian ring, that is, a commutative ring where any ideal $I \subset A$ is finitely generated. Show that $\operatorname{Spec} A$ is compact.

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