

EXERCISE SESSION 10

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EXERCISE 28.6

Let (X, d) be a metric space. If $f : X \rightarrow X$ satisfies the condition

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$, then f is called an isometry of X . Show that if f is an isometry and X is compact, then f is bijective and hence a homeomorphism.

Indeed, metric spaces are Hausdorff so a bijective map $f : X \rightarrow X$ is a homeomorphism if and only if it is bijective and continuous. To show that $f : X \rightarrow X$ is continuous note that given $\epsilon > 0$ and $x \in X$ we can pick $\delta = \epsilon$ since

$$d(x, y) < \epsilon \implies d(f(x), f(y)) = d(x, y) < \epsilon.$$

Injectivity of f is also easy to check: if $f(x) = f(y)$ then

$$0 = d(f(x), f(y)) = d(x, y) \implies x = y.$$

Surjectivity is the trick part. Let us assume contradictory that there is some $a \notin f(X)$ and choose $\epsilon > 0$ such that the ϵ -ball $B_d(a, \epsilon)$ is disjoint from $f(X)$. Now consider the sequence $(f^n(a))_{n=1}^\infty$. Note that we must have $d(x_n, x_m) \geq \epsilon$ for $n \neq m$. Indeed, without loss of generality we assume that $n < m$, and then

$$d(x_n, x_m) = d(f^n(a), f^m(a)) = d(a, f^{m-n}(a)) \geq \epsilon$$

since the ϵ -ball around a was disjoint from $f(X)$. Hence, the sequence $(f^n(a))_{n=1}^\infty$ has no convergent subsequence, contradicting the compactness of X .

EXERCISE 34.3

Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

Assume that X is metrizable. We showed in Problem 1 from Exercise Session 7 that compact metric spaces have countable dense subsets, and then in Problem 2 that metric spaces with countable dense subsets has a countable basis for its topology.

Conversely, note that compact Hausdorff spaces are normal by [Mun00, Theorem 32.3], hence regular. So if X is second countable then Urysohn's metrization theorem [Mun00, Theorem 34.1] implies that X is metrizable.

EXERCISE 35.9

Let $X_1 \subset X_2 \subset \dots$ be a sequence of spaces, where X_i is a closed subspace of X_{i+1} for each i . Let X be the union of the X_i ; let us topologize X by declaring a set U to be open in X if $U \cap X_i$ is open in X_i for each i .

- (a) Show that this is a topology on X and that each space X_i is a subspace (in fact, a closed subspace) of X in this topology. This topology is called the topology coherent with the subspaces X_i .
- (b) Show that $f : X \rightarrow Y$ is continuous if $f|_{X_i} : X_i \rightarrow Y$ is continuous for each i .
- (c) Show that if each space X_i is normal, then X is normal.

- (a) Let us check the three conditions of the first Definition in [Mun00, Section 12]:
 - (1) The empty set $\emptyset$ is open in X since $\emptyset \cap X_i = \emptyset$ is open in X_i for all i . Similarly, the set X itself is open since $X \cap X_i = X_i$ is open in X_i for all i .
 - (2) Let $\{U_\alpha\}$ be a collection of open sets in X . Given some fixed subspace X_i we have

$$\left(\bigcup_{\alpha} U_{\alpha} \right) \cap X_i = \bigcup_{\alpha} U_{\alpha} \cap X_i.$$

Since $U_{\alpha} \cap X_i$ is open in X_i for all α it follows that the union is open in X_i . Hence the union is open in X .

- (3) Let $\{U_j\}_{j=1}^n$ be a finite collection of open sets in X . Given some fixed subspace X_i we have

$$\left(\bigcap_{j=1}^n U_j \right) \cap X_i = \bigcap_{j=1}^n U_j \cap X_i.$$

Since $U_j \cap X_i$ is open in X_i for all j it follows that the intersection is open in X_i . Hence the intersection is open in X .

- (b) Assume that $f|_{X_i} : X_i \rightarrow Y$ is continuous for all i . Let U be an open set in Y . We have that

$$f|_{X_i}^{-1}(U) = f^{-1}(U) \cap X_i$$

is open in X_i for all i , because of continuousness of these maps. This means that $f^{-1}(U)$ is open in X , that is, f is continuous.

- (c) Let A and B be disjoint closed sets in X . We will prove that X is normal by finding a continuous map $f : X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. As preparation for this we start by inductively constructing continuous maps $f_i : A \cup B \cup X_i \rightarrow [0, 1]$ satisfying the above as well.

Let us start with the case $n = 1$. Since X_1 is normal we can by Urysohn's lemma [Mun00, Theorem 33.1] find a continuous map $g_1 : X_1 \rightarrow [0, 1]$ such that

$$f^{-1}(A \cap X_1) = \{0\} \quad \text{and} \quad f^{-1}(B \cap X_1) = \{1\}.$$

We define the map $f_1 : A \cup B \cup X_1 \rightarrow [0, 1]$ by setting

$$f_1(x) = \begin{cases} 0 & \text{if } x \in A \\ g_1(x) & \text{if } x \in X_1 \\ 1 & \text{if } x \in B \end{cases}$$

This map is continuous by the pasting lemma [Mun00, Lemma 18.3].

With the base case in place, we construct $f_{i+1} : A \cup B \cup X_{i+1} \rightarrow [0, 1]$ from $f_i : A \cup B \cup X_i \rightarrow [0, 1]$ in the following fashion. Firstly, consider the subset $(A \cup B \cup X_i) \cap X_{i+1}$ of X_{i+1} ; this is closed in the normal space X_{i+1} . We conclude that by Tietze's extension theorem [Mun00, Theorem 35.1] the restricted map $f_i : (A \cup B \cup X_i) \cap X_{i+1} \rightarrow [0, 1]$ can be extended to a continuous map

$$g_{i+1} : X_{i+1} \rightarrow [0, 1].$$

We define $f_{i+1} : A \cup B \cup X_{i+1} \rightarrow [0, 1]$ by setting

$$f_{i+1}(x) = \begin{cases} 0 & \text{if } x \in A \\ g_{i+1}(x) & \text{if } x \in X_{i+1} \\ 1 & \text{if } x \in B \end{cases} .$$

This is again continuous by the pasting lemma [Mun00, Lemma 18.3].

With all the continuous maps $f_i : A \cup B \cup X_i \rightarrow [0, 1]$ in place, we are now ready to define $f : X \rightarrow [0, 1]$. For $x \notin A \cup B$, let

$$n_x = \min\{n \mid x \in X_n\} .$$

We define

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ f_{n_x}(x) & \text{if } x \notin A \cup B \\ 1 & \text{if } x \in B \end{cases}$$

We note that this is well-defined, and that $f|_{X_i} = g_i : X_i \rightarrow [0, 1]$ is continuous for all i . The map f is hence continuous by (b).

EXERCISE 43.1

Let X be a metric space.

- (a) Suppose that for some $\epsilon > 0$, every ϵ -ball in X has compact closure. Show that X is complete.
- (b) Suppose that for each $x \in X$ there is an $\epsilon > 0$ such that the ball $B(x, \epsilon)$ has compact closure. Show by means of an example that X need not be complete.

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- (1) Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence. It is enough to show that this has a convergent subsequence, since completeness of X then follows from [Mun00, Lemma 43.1]. Given $\epsilon > 0$ as in the statement of the problem, we know that there is some N such that $d(x_m, x_n) < \epsilon$ whenever $n, m \geq N$. We also know that the closure of the open ball $B_d(x_N, \epsilon)$ is compact, and that it contains all but possibly the first $N - 1$ terms in the sequence. This implies by [Mun00, Theorem 28.2] that the sequence has a convergent subsequence.
 - (2) Consider the submetric space $K = \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$ with its standard metric. Indeed, for every $1/n$ we can find an $\epsilon > 0$ so that $B_d(1/n, \epsilon) = \{1/n\}$ which is closed in K , so taking its closure doesn't do anything. It is also a compact set. Note, however that K is not a complete metric space. Indeed, consider the Cauchy sequence $(1/n)_{n=1}^\infty$ in K . This does not converge to any point in K .

EXERCISE 43.2

Let (X, d_X) and (Y, d_Y) be metric spaces; let Y be complete. Let $A \subset X$. Show that if $f : A \rightarrow Y$ is uniformly continuous, then f can be uniquely extended to a continuous function $g : \bar{A} \rightarrow Y$, and g is uniformly continuous.

Uniqueness: Note that uniqueness of the extension follows from Exercise 18.13, since Y being a metric space implies that it is Hausdorff.

Existence: We now define g in a reasonable way. Let $x \in \bar{A}$; since X is metrisable we know that there is a sequence $(x_n)_{n=1}^\infty$ in A converging to x by [Mun00, Lemma 21.1]. We claim that the sequence $(f(x_n))_{n=1}^\infty$ converges in Y ; since Y is complete it suffices to show that it is a Cauchy sequence. Given $\epsilon > 0$, uniform continuity of f implies that there is an $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

Now, since $(x_n)_{n=1}^\infty$ is convergent it is a Cauchy sequence, so there is an N such

$$d_X(x_n, x_m) < \delta \quad \text{whenever } m, n \geq N.$$

This proves that $(f(x_n))_{n=1}^\infty$ is Cauchy, and hence converges. Let us set

$$g(x) = \lim_{n \rightarrow \infty} f(x_n).$$

Uniform continuity: The last thing we need to show is that the map

$$g : \bar{A} \rightarrow Y, \quad x \mapsto \lim_{n \rightarrow \infty} f(x_n)$$

is uniformly continuous. Let us pick $\epsilon > 0$; we want to show that there is some $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(g(x), g(y)) < \epsilon.$$

Since f is uniformly continuous we know there is some $\delta_f > 0$ such that

$$d_X(x, y) < \delta_f \implies d_Y(f(x), f(y)) < \epsilon/3.$$

We claim that we can pick $\delta = \delta_f/3$. To prove this, let us pick points x and y such that $d_X(x, y) < \delta_f/3$. Firstly, since $(x_n)_{n=1}^\infty$ converges to x and $(y_n)_{n=1}^\infty$ converges to y we can find some N such that

$$d_X(x_n, x) < \delta_f/3 \quad \text{and} \quad d_X(y_n, y) < \delta_f/3$$

whenever $n \geq N$. In this case, we can use the triangle inequality to make sure that

$$d_X(x_n, y_n) \leq d_X(x_n, x) + d_X(x, y) + d_X(y, y_n) < \delta_f,$$

whenever $n \geq N$. This then implies that $d_X(f(x_n), f(y_n)) < \epsilon/3$. Secondly, the fact that $(f(x_n))_{n=1}^\infty$ and $(f(y_n))_{n=1}^\infty$ converges to $g(x)$ and $g(y)$, respectively, guarantees that there is some M such that

$$d_Y(f(x_n), g(x)) < \epsilon/3 \quad \text{and} \quad d_X(f(y_n), g(y)) < \epsilon/3$$

whenever $n \geq M$. Now, taking $n \geq \max\{N, M\}$ we obtain

$$d_Y(g(x), g(y)) \leq d_Y(g(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(f(y_n), g(y)) < \epsilon,$$

which proves that g is uniformly continuous.

EXERCISE 43.3

Two metrics d and d' on a set X are said to be metrically equivalent if the identity map $i : (X, d) \rightarrow (X, d')$ and its inverse are both uniformly continuous.

- (a) Show that d is metrically equivalent to the standard bounded metric $\bar{d}$ derived from d .
- (b) Show that if d and d' are metrically equivalent, then X is complete under d if and only if it is complete under d' .

- (a) Recall that the bounded metric derived from d is the metric given as

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

We want to show that the identity map $\text{id}_X : (X, d) \rightarrow (X, \bar{d})$ is uniformly continuous. Indeed, given $0 < \epsilon < 1$ we can pick $\delta = \epsilon/2$ since

$$d(x, y) < \epsilon/2 \implies \bar{d}(x, y) < \epsilon/2 < \epsilon$$

Similarly, the identity map $\text{id}_X : (X, \bar{d}) \rightarrow (X, d)$ going the other way around is uniformly continuous. Indeed, given $0 < \epsilon < 1$ we can pick $\delta = \epsilon$ since

$$\bar{d}(x, y) < \epsilon \implies d(x, y) < \epsilon.$$

- (b) Assume that X is complete under d . We want to show that it is also complete under d' . Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in (X, d') ; we claim that this is also a Cauchy sequence in (X, d) . Indeed, pick $\epsilon > 0$. We want to show that there is an N such that

$$d(x, y) < \epsilon$$

whenever $m, n \geq N$. Because of uniform continuousness of the map $\text{id} : (X, d') \rightarrow (X, d)$ we know that there is some $\delta > 0$ such that

$$d'(x, y) < \delta \implies d(x, y) < \epsilon,$$

for all $x, y \in X$. Since $(x_n)_{n=1}^\infty$ is a Cauchy sequence in (X, d') we can now find N such that

$$d'(x_n, x_m) < \delta$$

whenever $n, m \geq N$. Then we have

$$d(x_n, x_m) < \epsilon$$

whenever $n, m \geq N$, which proves that the sequence is also Cauchy in (X, d) . Since (X, d) is complete we know that it has a limit, which must also be the limit in (X, d') by continuousness.

EXERCISE 43.10

Theorem 1. *Let $h : X \rightarrow Y$ and $h' : X \rightarrow Y'$ be isometric imbeddings of the metric space (X, d) in the complete metric spaces (Y, D) and (Y', D') , respectively. Then there is an isometry of $(\overline{h(X)}, D)$ with $(\overline{h'(X)}, D')$ that equals $h'h^{-1}$ on the subspace $h(X)$.*

We are given $h : X \rightarrow Y$ and $h' : X \rightarrow Y'$ such that

$$d(x, y) = D(h(x), h(y)) \quad \text{and} \quad d(x, y) = D'(x, y)$$

whose restrictions to image becomes a homeomorphism; that is, we have homeomorphisms $h : X \cong h(X)$ and $h' : X \cong h'(X)$. Consider the composition

$$h(X) \xrightarrow{h^{-1}} X \xrightarrow{h'} Y'.$$

Of course, we could restrict the target a bit and consider it as a map $\varphi : h(X) \rightarrow \overline{h'(X)}$. Note that this is also an isometric imbedding. Note also that isometries are always uniformly continuous and that closed subsets of complete metric spaces are also complete, so by Exercise 43.2 this map can be uniquely extended to continuous map $\psi : \overline{h(X)} \rightarrow \overline{h'(X)}$. We claim that this map is necessarily an isometry. Indeed, recall that the unique extension ψ was constructed so that

$$\psi(x) = \lim_{n \rightarrow \infty} \varphi(x_n)$$

where $(x_n)_{n=1}^{\infty}$ is a sequence in $h(X)$ converging to x . Since the map φ is an isometry we have

$$D(x_n, y_n) = D'(\varphi(x_n), \varphi(y_n)),$$

and letting $n \rightarrow \infty$ we obtain

$$D(x, y) = D'(\psi(x), \psi(y)),$$

which proves that also ψ is an isometry.

REFERENCES

[Mun00] J. R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.