

EXERCISE SESSION 11

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PROBLEM 1

Let M be the subspace of $\mathrm{SO}(3)$ consisting of matrices A such that $A^2 = I$ and $A \neq I$, where I is the identity matrix. Show that M is homeomorphic to $\mathbb{RP}^2$.

The condition $A^2 = I$ on $A \in \mathrm{SO}(3)$ just tells us that A is a symmetric matrix, that is, that $A = A^T$. So M is then all “non-identity symmetric orthogonal 3×3 -matrices with determinant equal to 1”. Let us compute the inverse image of M under the map $\mathrm{Ad} : \mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$ defined in the lecture notes [Frø19]. What follows will resemble Theorem 4.1 in the mentioned reference. We know that any matrix $A \in \mathrm{SO}(3)$ is a rotation around some axis in $\mathbb{R}^3$, so we can write it as

$$A = C \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} C^T$$

for some matrix $C \in \mathrm{SO}(3)$ and angle $0 \leq \theta < 2\pi$. Note that A is symmetric if and only if the cosine-sine rotation matrix is symmetric. That is, if and only if

$$\sin \theta = -\sin \theta \iff \theta = 0, \pi.$$

If $\theta = 0$, then we are just dealing with the identity matrix and we didn’t want to consider that one. So we focus on the case $\theta = \pi$. As in the lecture notes we have that

$$\mathrm{Ad} \left(\cos \left(\frac{\theta}{2} \right) \cdot 1 + \sin \left(\frac{\theta}{2} \right) \cdot w \right) = A$$

where w is the third column of C . Note that the columns of C are of length 1 (by virtue of C being an orthogonal matrix), so we can think of Ad , where we restrict the target to M , as a continuous map

$$\mathrm{Ad} : S^2 \rightarrow M.$$

We note that we have $\mathrm{Ad}(w) = \mathrm{Ad}(q)$ if and only if $q = \pm w$, so dividing out by the antipodal action of $\{\pm 1\}$ on the sphere we get a homeomorphism

$$\mathbb{RP}^2 \cong S^2 / \{\pm 1\} \cong M.$$

PROBLEM 2

A linear map $T : E \rightarrow F$ between normed vector spaces is called a compact operator if the closure of $T(A)$ is compact for every bounded subset $A \subset E$. Show that if

$$k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$$

is continuous then the operator

$$T : \mathcal{C}([0, 1]) \rightarrow \mathcal{C}([0, 1])$$

given by

$$(Tf)(x) = \int_0^1 k(x, y)f(y)dy$$

is compact. (Here $\mathcal{C}([0, 1])$ denotes the space of continuous real-valued functions on the unit interval, equipped with the supremum norm.)

Let A be a bounded subset of $\mathcal{C}([0, 1])$. That is, A is a collection of continuous maps $f : [0, 1] \rightarrow \mathbb{R}$ such that there is some M such that

$$|f| := \sup_{x \in I} |f(x)| \leq M$$

for all $f \in A$. We want to show that the closure of $T(A)$ in $\mathcal{C}([0, 1])$ is compact. By Ascoli's theorem [Mun00, Theorem 45.4] it suffices to show that $T(A)$ is equicontinuous and pointwise bounded.

Pointwise bounded: Note that $k : I^2 \rightarrow \mathbb{R}$ bounded since I^2 compact. That is, there is some N such that

$$|k(x, y)| \leq N$$

for all $(x, y) \in I^2$. Then

$$\begin{aligned} |(Tf)(x)| &= \left| \int_0^1 k(x, y)f(y)dy \right| \\ &\leq \int_0^1 |k(x, y)||f(y)|dy \\ &\leq \int_0^1 N \cdot M dy \\ &\leq N \cdot M \end{aligned}$$

which shows that Tf is boundwise bounded.

Equicontinuous: Given $\epsilon > 0$ we want to find $\delta > 0$ such that

$$|x - x'| < \delta \implies |(Tf)(x) - (Tf)(x')| < \epsilon$$

for all $f \in A$. We first note that k is uniformly continuous by Theorem 27.6. Hence, there is some $\delta > 0$ such that

$$|(x, y) - (x', y')| < \delta \implies |k(x, y) - k(x', y')| < \epsilon/M.$$

We claim that we use this δ to show equicontinuity of $T(A)$. Indeed, we have

$$\begin{aligned} |(Tf)(x) - (Tf)(x')| &= \left| \int_0^1 k(x, y) - k(x', y)f(y)dy \right| \\ &\leq \int_0^1 |k(x, y) - k(x', y)||f(y)|dy \\ &< \int_0^1 \frac{\epsilon}{M} M dy \\ &\leq \epsilon \end{aligned}$$

whenever $|x - x'| < \delta$, which shows equicontinuity.

EXERCISE 21.9

Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the zero function.

- (a) Show that $f_n(x) \rightarrow f(x)$ for each $x \in \mathbb{R}$.
- (b) Show that f_n does not converge uniformly to f .

- (a) Fixing x we see that

$$\lim_{n \rightarrow \infty} (n^3[x - (1/n)]^2 + 1) = \left(\lim_{n \rightarrow \infty} n^3 \right) \cdot \left(\lim_{n \rightarrow \infty} (x - (1/n))^2 \right) + 1 = \infty,$$

so we conclude that

$$\lim_{n \rightarrow \infty} f_n(x) = 0.$$

- (b) Let us pick $\epsilon = 1/2$. If the convergence was uniform, then there would be a positive integers N such that

$$f_n(x) < 1/2$$

for all $n \geq N$ and $x \in X$. This would in particular mean that

$$n^3[x - (1/n)]^2 > 1$$

for all $n \geq N$ and $x \in X$. However, this is impossible since for $n = N$ and $x = 1/N$ the conclusion would be that

$$0 > 1.$$

EXERCISE 45.4

- (a) Let $f_n : I \rightarrow \mathbb{R}$ be the function $f_n(x) = x^n$. The collection $\mathcal{F} = \{f_n\}$ is pointwise bounded but the sequence (f_n) has no uniformly convergent subsequence; at what point or points does $\mathcal{F}$ fail to be equicontinuous?
- (b) Repeat (a) for the functions f_n of Exercise 9 of §21.

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- (a) The collection fails to be equicontinuous at the point $x_0 = 1$. Indeed, pick $\epsilon = 1/2$. If $\mathcal{F}$ was equicontinuous at $x_0 = 1$ this would mean that there was a neighborhood $(\delta, 1]$ around $x_0 = 1$ such that

$$|f_n(x_0) - f_n(x)| = 1 - x^n < 1/2$$

for all n and $\delta < x \leq 1$. This is of course impossible, since $x^n \rightarrow 0$ on the interval $[0, 1)$.

- (b) The collection fails to be equicontinuous at the points of the set

$$K = \{1/m \mid m \in \mathbb{Z}_+\}.$$

To show that the collection fails to be continuous at $x_0 = 1/m$, let us pick $0 < \epsilon < 1/m$. If $\mathcal{F}$ was equicontinuous at x_0 , then there would be some $\delta > 0$ such that

$$|x_0 - x| < \delta \implies |f_n(x_0) - f_n(x)| < \epsilon < 1/m$$

for all n . To get a contradiction let us pick $n = m$. Then this would say that

$$1 - \frac{1}{m^3[x - (1/m)]^2 + 1} < 1/m \iff m^3[x - 1/m]^2[m - 1] < 0$$

whenever $|x_0 - x| < \delta$. But this is impossible, since all of the factors on the left hand side of the inequality are non-negative.

EXERCISE 45.5

Let X be a space. A subset $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R})$ is said to vanish uniformly at infinity if given $\epsilon > 0$ there is a compact subspace C of X such that $|f(x)| < \epsilon$ for $x \in X \setminus C$ and $f \in \mathcal{F}$. If $\mathcal{F}$ consists of a single function f , we simply say that f vanishes at infinity. Let $\mathcal{C}_0(X, \mathbb{R})$ denote the set of continuous functions $f : X \rightarrow \mathbb{R}$ that vanish at infinity.

Theorem 1. *Let X be locally compact Hausdorff; give $\mathcal{C}_0(X, \mathbb{R})$ the uniform topology. A subset of $\mathcal{F}$ of $\mathcal{C}_0(X, \mathbb{R})$ has compact closure if and only if it is pointwise bounded, equicontinuous, and vanishes uniformly at infinity.*

We want to first embed $\mathcal{C}_0(X, \mathbb{R})$ isometrically into $\mathcal{C}(Y, \mathbb{R})$, where Y denotes the one-point compactification of X . The obvious way of defining the map $E : \mathcal{C}_0(X, \mathbb{R}) \rightarrow \mathcal{C}(Y, \mathbb{R})$ is by setting

$$(Ef)(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}.$$

This is well-defined; more specifically, $Ef : Y \rightarrow \mathbb{R}$ is a continuous map. Since we have $(Ef)|_X = f$, which is clearly continuous, we only need to show that Ef is continuous at ∞ . This follows from the fact that f was assumed to vanish at infinity. Indeed, given some neighborhood $(-\epsilon, \epsilon)$ around $(Ef)(\infty) = 0 \in \mathbb{R}$, we know from the fact that f is bounded at infinity that there is a compact subspace C of X such that

$$f(Y \setminus C) \subset (-\epsilon, \epsilon).$$

Of course, $Y \setminus C$ is a neighborhood of ∞ in the one-point compactification Y , so this shows that Ef is continuous by [Mun00, Theorem 18.1(4)]

Lemma 2. *The map $E : \mathcal{C}_0(X, \mathbb{R}) \rightarrow \mathcal{C}(Y, \mathbb{R})$ defined above is an isometry with closed image.*

Proof. We start by showing that Ef is an isometry. First note that

$$\sup_{y \in Y} |(Ef)(y) - (Eg)(y)| = \sup_{x \in X} |f(x) - g(x)|$$

which directly implies that they agree also on the uniform metric. Next, we show that the image of E is closed in $\mathcal{C}(Y, \mathbb{R})$. The claim is that we have

$$\text{Im}(E) = \{g : Y \rightarrow \mathbb{R} \mid g(\infty) = 0\}.$$

The right hand side is a closed set since it is the inverse image of the diagonal in $\mathbb{R}$ under the continuous map

$$\varphi : \mathcal{C}(Y, \mathbb{R}) \xrightarrow{(\text{ev}_\infty, 0)} \mathbb{R} \times \mathbb{R}$$

where ev_∞ denotes evaluation at ∞ .¹ One of the inclusions is clear. For the other one we need to show that if $g : Y \rightarrow \mathbb{R}$ is a continuous map such that $g(\infty) = 0$, then the restricted map $g|_X : X \rightarrow \mathbb{R}$ vanishes at infinity. So let us pick $\epsilon > 0$. Since g is continuous, it is in particular continuous at ∞ . Since $g(\infty) = 0$ we know that we can find an open U containing ∞ such that

$$g(U) \subset (-\epsilon, \epsilon).$$

Open sets containing ∞ are on the form $U = Y \setminus C$ where C is a compact set in X . This tells us that

$$g|_X(X \setminus C) \subset (-\epsilon, \epsilon)$$

which means that $g|_X$ vanishes at infinity. □

Corollary 3. *A collection $\mathcal{F}$ has compact closure in $\mathcal{C}_0(X, \mathbb{R})$ if and only if $E(\mathcal{F})$ has compact closure in $\mathcal{C}(Y, \mathbb{R})$.*

¹Evaluation is a continuous map by [Mun00, Theorem 46.10]. Here we are using that Y is compact to identify the compact-open topology on $\mathcal{C}(Y, \mathbb{R})$ with the uniform topology on the same set. See also [Mun00, Exercise 43.8].

Proof. Isometries are embeddings so we know from the previous lemma that there is a homeomorphism

$$\mathcal{C}_0(X, \mathbb{R}) \cong \text{Im}(E).$$

Hence, the closure of $\mathcal{F}$ in the left hand side is equal to the closure of $E(\mathcal{F})$ in the right hand side. Since the right hand side is closed in $\mathcal{C}(Y, \mathbb{R})$ this equals the closure of $E(\mathcal{F})$ in $\mathcal{C}(Y, \mathbb{R})$. $\square$

We also need the following result.

Corollary 4. *A collection $\mathcal{F}$ in $\mathcal{C}_0(X, \mathbb{R})$ is equicontinuous and vanishes uniformly at infinity if and only if $E(\mathcal{F})$ is equicontinuous in $\mathcal{C}(Y, \mathbb{R})$.*

Proof. It is straight forward to see that $\mathcal{F}$ is equicontinuous at $x_0 \in X$ if and only if $E(\mathcal{F})$ is equicontinuous at the same point, but now regarded as a point in $Y = X \cup \{\infty\}$. We claim that $E(\mathcal{F})$ is equicontinuous at $x_0 = \infty$ if and only if $\mathcal{F}$ vanishes uniformly at infinity. Let $\epsilon > 0$. We know that $\mathcal{F}$ vanishes uniformly at infinity if and only if there is a compact subspace C of X such that

$$f(X \setminus C) \subset (-\epsilon, \epsilon),$$

for all $f \in \mathcal{F}$. So setting $U = Y \setminus C$ shows that $E(\mathcal{F})$ is equicontinuous at $x_0 = \infty$. The other implication is similar. $\square$

We now put these results together to prove the theorem.

By Corollary 3, $\mathcal{F}$ has compact closure in $\mathcal{C}_0(X, \mathbb{R})$ if and only if $E(\mathcal{F})$ has compact closure in $\mathcal{C}(Y, \mathbb{R})$. By Ascoli's theorem, this happens if and only if $E(\mathcal{F})$ is pointwise bounded and equicontinuous in $\mathcal{C}(Y, \mathbb{R})$. By Corollary 4, this happens if and only if $\mathcal{F}$ is pointwise bounded, equicontinuous, and vanishes uniformly at infinity.

EXERCISE 48.10

Let X be a complete metric space, and let $\mathcal{F}$ be a subset of $\mathcal{C}(X, \mathbb{R})$ such that for each $a \in X$, the set

$$\mathcal{F}_a = \{f(a) \mid f \in \mathcal{F}\}$$

is bounded. Then there is a nonempty open set U of X on which the functions in $\mathcal{F}$ are uniformly bounded, that is, there is a number M such that $|f(x)| \leq M$ for all $x \in U$ and all $f \in \mathcal{F}$.

We first note that complete metric spaces are Baire spaces [Mun00, Theorem 48.2]. For positive integers N let us define

$$A_N = \{x \in X \mid |f(x)| \leq N \text{ for all } f \in \mathcal{F}\}.$$

We note that these sets are closed in X since we can write

$$A_N = \bigcap_{f \in \mathcal{F}} f^{-1}([-N, N])$$

and all the f are continuous. Because of pointwise boundedness of the collection $\mathcal{F}$ we can always find some positive integer M_a such that $|f(a)| \leq M_a$ for all $f \in \mathcal{F}$. We conclude that

$$X = \bigcup_{N \geq 1} A_N.$$

Since X is a Baire space one of the A_N must have non-empty interior, so it is possible to find an open ball in one of them. In other words, we can find a point $x_0 \in X$ and some $\epsilon > 0$ such that

$$U := B_d(x_0, \epsilon) \subset A_{N_0}$$

where N_0 is some positive integer. But then $\mathcal{F}$ is uniformly bounded on U , since $x \in U$ implies that $x \in A_{N_0}$ so that

$$|f(x)| \leq N_0$$

for all $f \in \mathcal{F}$.

REFERENCES

- [Frø19] K. A. Frøysov, *Matrix groups* (2019). <https://www.uio.no/studier/emner/matnat/math/MAT4500/h19/beskjeder/matrix.pdf>.
- [Mun00] J. R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.