

EXERCISE SESSION 12

ALICE HEDENLUND

EXERCISE 45.1

If X_n is metrizable with metric d_n , then

$$D(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}_i(x_i, y_i)/i\}$$

is a metric for the product space $X = \prod_n X_n$. Show that X is totally bounded under D if each X_n is totally bounded under d_n . Conclude without using the Tychonoff theorem that a countable product of compact metrizable spaces is compact.

If the X_n are compact, then they are complete and totally bounded under d_n by [Mun00, Theorem 54.1]. We want to prove that X is totally bounded and complete under D , since this would imply that X is compact, again by [Mun00, Theorem 54.1].

Total boundedness: Let us fix $\epsilon > 0$. There is some N such that $\epsilon n > 1$ for all $n \geq N$. For $1 \leq n < N$, since X_n is totally bounded under d_n we know that we can find a finite set F_n of points in X_n such that

$$X_n = \bigcup_{x \in F_n} B_{d_n}(x, n\epsilon).$$

For $n \geq N$ we pick any point z_n in X_n . Let us set

$$F = \prod_{n=1}^{N-1} F_n \times \prod_{n \geq N} \{z_n\}.$$

This is a finite set of points in X and we claim that

$$X = \bigcup_{\mathbf{x} \in F} B_D(\mathbf{x}, \epsilon).$$

Indeed, if $\mathbf{y}$ is some arbitrary point in X and consider the ϵ -ball

$$\begin{aligned} B_D(\mathbf{y}, \epsilon) &= \{\mathbf{x} \in X \mid D(\mathbf{x}, \mathbf{y}) < \epsilon\} \\ &= \{\mathbf{x} \in X \mid \sup_n \{\bar{d}_n(x_n, y_n)/n\} < \epsilon\} \end{aligned}$$

In particular this tells us that

$$\bar{d}_n(x_i, y_i) < n\epsilon,$$

and by construction F is a set which contains such an $\mathbf{x}$.

Completeness: Let $(\mathbf{x}_i)_{i=1}^\infty$ be a Cauchy sequence in (X, D) . We start by claiming that $(\pi_n(\mathbf{x}_i))_{i=1}^\infty$ is a Cauchy sequence in $(X_n, \bar{d}_n)$. Indeed, let us fix $\epsilon > 0$. Since $(\mathbf{x}_i)_{i=1}^\infty$ is a Cauchy sequence in (X, d) there is some I_n such that

$$D(\mathbf{x}_i, \mathbf{x}_j) < \epsilon/n$$

for $i, j \geq I_n$. Then we have

$$\bar{d}_n(\pi_n(\mathbf{x}_i), \pi_n(\mathbf{x}_j)) \leq nD(\mathbf{x}_i, \mathbf{x}_j) < \epsilon$$

for $i, j \geq I_n$, which shows that the sequence is Cauchy in $(X_n, \bar{d}_n)$. Since (X_n, d_n) is complete, we know that $(X_n, \bar{d}_n)$ is complete, so that $(\pi_n(x_i))_{i=1}^\infty$ converges to some point y_n in X_n . Then the sequence $(\mathbf{x}_i)_{i=1}^\infty$ converges to the point $\mathbf{y} := (y_n)_{n=1}^\infty$ by Exercise 19.6.

EXERCISE 46.4

Consider the sequence of continuous functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f_n(x) = x/n.$$

In which of the three topologies of Theorem 46.7 does this sequence converge? Answer the same question for the sequence given in Exercise 9 of §21.

We start with the sequence above.

Uniform convergence: We do not have uniform convergence. For example, let $\epsilon = 1/2$. If we had uniform convergence we would be able to find an integer N such that

$$\frac{|x|}{n} < \frac{1}{2}$$

for $n \geq N$ and all $x \in \mathbb{R}$. However, picking $n = N$, we can just pick $x > N/2$ to get a contradiction.

Compact convergence: We have compact convergence. Indeed, let C be a compact subspace of $\mathbb{R}$. Since compact subspaces in $\mathbb{R}$ are bounded, we know that $|x| < M$ for some integer M for $x \in C$. Given $\epsilon > 0$, we can choose N to be some integer larger than M/ϵ . Then

$$\frac{|x|}{n} < \epsilon$$

for all $n \geq N$ and $x \in C$.

Pointwise convergence: Pointwise convergence follows from compact convergence.

Now we consider the sequence of functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ appearing in Exercise 21.9, namely

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

The goal of that exercise was proving that this sequence converges pointwise, but not uniformly, so the only new case is compact convergence.

Uniform convergence: We do not have uniform convergence. See Exercise 21.9.

Compact convergence: We do not have compact convergence. Indeed, consider the compact set

$$K = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\},$$

and pick $\epsilon = 1/2$. If we had compact convergence we would be able to find N such that

$$\frac{1}{n^3[x - (1/n)]^2 + 1} < \frac{1}{2} \implies 1 < n^3[x - (1/n)]^2$$

for all $n \geq N$ and $x \in K$. But pick $n = N$ and $x = 1/N$. This would then tell us that

$$1 < 0$$

which is a contradiction.

Pointwise convergence: We have pointwise convergence. See Exercise 21.9.

EXERCISE 46.6

Show that in the compact-open topology, $\mathcal{C}(X, Y)$ is Hausdorff if Y is Hausdorff, and regular if Y is regular.

Hausdorff: Let $f \neq g$ be two continuous functions $X \rightarrow Y$, where Y is assumed to be Hausdorff. Then we have $f(x) \neq g(x)$ for at least some $x \in X$, and since Y is Hausdorff we find disjoint neighborhoods U and V of $f(x)$ and $g(x)$, respectively. Then the subbasis elements $S(\{x\}, U)$ and $S(\{x\}, V)$ are disjoint open sets containing f and g , respectively, which shows that $\mathcal{C}(X, Y)$ is Hausdorff.

Regular: Let $S(C, U)$ be a subbasis element in $\mathcal{C}(X, Y)$, where Y is assumed to be regular. We will use the characterisation of regularity as in [Mun00, Lemma 31.1]. We know that we can find an open set V in Y such that $\overline{V} \subset U$. We claim that

$$\overline{S(C, V)} \subset S(C, U),$$

which directly implies that $\mathcal{C}(X, Y)$ is regular. To show the claim let $f \in \overline{S(C, V)}$. This means that any subbasis element $S(K, W)$ containing f intersects $S(C, V)$. In other words: if K is a compact subspace of X and W is an open set of Y such that $f(K) \subset W$, then we can find some continuous map $g : X \rightarrow Y$ such that

$$g(K) \subset W \quad \text{and} \quad g(C) \subset V.$$

We want to show that this implies that $f(C) \subset U$. Indeed, assume contradictory that it was not so. Then we could find some $x \in C$ such that $f(x) \notin U$. Since $\overline{V} \subset U$ this would mean that $f(x) \notin \overline{V}$. With this in mind let us set

$$K = \{x\} \quad \text{and} \quad W = Y \setminus \overline{V}.$$

By the above discussion we can then find a continuous map $g : X \rightarrow Y$ such that

$$g(x) \notin \overline{V} \quad \text{and} \quad g(C) \subset V.$$

But x was an element of C so this tells us that $g(x) \in V$ and $g(x) \notin \overline{V}$, which is the wanted contradiction.

EXERCISE 46.7

Show that if Y is locally compact Hausdorff, then composition of maps

$$c : \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$$

is continuous, provided the compact-open topology is used throughout.

Consider a subbasis element $S(C, U)$ containing the composition $c(f, g) = g \circ f : X \rightarrow Z$, where $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous maps. We note that $f(C)$ is a compact set contained in the open set $g^{-1}(U)$ of Y . We are going to need the following lemma.

Lemma 1. *Let Y be locally compact Hausdorff. If K is a compact subspace of Y and W is an open set containing K , then there is an open set V containing K such that $\bar{V}$ is compact and $\bar{V} \subset W$.*

Proof. Let us apply [Mun00, Theorem 29.2] for every point in K . That is, for all $x \in K$ there is a neighborhood V_x of x such that $\bar{V}_x$ is compact and $\bar{V}_x \subset W$. Then the collection $\{V_x\}$ forms an open cover of K , and since K is compact we know that there is a finite subcover. That is, we can find points $x_1, \dots, x_n$ in K such that

$$V := \bigcup_{i=1}^n V_{x_i} \supset K.$$

We note that $\bar{V} \subset W$ and that $\bar{V}$ is compact since it is a finite union of compact sets. □

By the above lemma we find an open V containing the compact set $f(C)$ such that $\bar{V} \subset g^{-1}(U)$ and $\bar{V}$ is compact. In conclusion, we have found an open set $S(C, V)$ containing f and another open set $S(\bar{V}, U)$ containing g , such that

$$c(S(C, V) \times S(\bar{V}, U)) \subset S(C, U)$$

which shows that c is continuous.

EXERCISE 46.8

Let $\mathcal{C}'(X, Y)$ denote the set $\mathcal{C}(X, Y)$ in some topology $\mathcal{T}$, Show that if the evaluation map

$$e : X \times \mathcal{C}'(X, Y) \rightarrow Y$$

is continuous, then $\mathcal{T}$ contains the compact-open topology.

Consider the induced map $E : \mathcal{C}'(X, Y) \rightarrow \mathcal{C}(X, Y)$ which is continuous by [Mun00, Theorem 46.11]. Note that the induced map is given by

$$(Ef)(x) = e(x, f) = f(x) \implies E = \text{id} : \mathcal{C}'(X, Y) \rightarrow \mathcal{C}(X, Y).$$

By Exercise 18.3 we conclude that $\mathcal{T}$ is finer than the compact-open topology.

EXERCISE 46.9

Theorem 2. *If $p : A \rightarrow B$ is a quotient map and X is locally compact Hausdorff, then $i_X \times p : X \times A \rightarrow X \times B$ is a quotient map.*

- Proof.* (a) Let Y be the quotient space induced by $i_X \times p$; let $q : X \times A \rightarrow Y$ be the quotient map. Show there is a bijective continuous map $f : Y \rightarrow X \times B$ such that $f \circ q = i_X \times p$.
(b) Let $g = f^{-1}$. Let $G : B \rightarrow \mathcal{C}(X, Y)$ and $Q : A \rightarrow \mathcal{C}(X, Y)$ be the maps induced by g and q , respectively. Show that $Q = G \circ p$.
(c) Show that Q is continuous; conclude that G is continuous, so that g is continuous. □

- (1) This is [Mun00, Corollary 22.3]. We note that in order to show that $i_X \times p : X \times A \rightarrow X \times B$ is a quotient map we need to show that the inverse of f is continuous, as well.
(2) The maps $G : B \rightarrow \mathcal{C}(X, Y)$ and $Q : A \rightarrow \mathcal{C}(X, Y)$ are defined as

$$(G(b))(x) = g(x, b) \quad \text{and} \quad (Q(a))(x) = q(x, a).$$

We note that since $f \circ q = i_X \times p$ it follows that $q = g \circ (i_X \times p)$, so that we have

$$\begin{aligned} ((G \circ p)(a))(x) &= (G(p(a)))(x) \\ &= g(x, p(a)) \\ &= q(x, a) \\ &= (Q(a))(x). \end{aligned}$$

- (3) Since q is continuous it follows that Q is continuous by [Mun00, Theorem 46.11]. The map G is then continuous since we have $Q = G \circ p$ where p is a quotient map. Since X is locally compact Hausdorff we can use [Mun00, Theorem 46.11] again to conclude that g is continuous.

REFERENCES

[Mun00] J. R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.