

EXERCISE SESSION 13

ALICE HEDENLUND

EXERCISE 51.1

Show that if $h, h' : X \rightarrow Y$ are homotopic and $k, k' : Y \rightarrow Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic

Let $H : X \times I \rightarrow Y$ and $K : Y \times I \rightarrow Z$ be homotopies between h and h' and k and k' , respectively. That is, H is a continuous map such that

$$H(x, 0) = h(x) \quad \text{and} \quad H(x, 1) = h'(x),$$

and similarly for K . We claim that the composition

$$X \times I \xrightarrow{(H, \pi_2)} Y \times I \xrightarrow{K} Z$$

is a homotopy between $k \circ h$ and $k' \circ h'$. Indeed, this map is continuous, since it is built by continuous maps and we have

$$(K \circ (H, \pi_2))(x, 0) = K(h(x), 0) = k(h(x)) \quad \text{and} \quad (K \circ (H, \pi_2))(x, 1) = K(h'(x), 1) = k'(h'(x)).$$

EXERCISE 51.2

Given spaces X, Y let $[X, Y]$ denote the set of homotopy classes of maps of X into Y .

- (a) Let $I = [0, 1]$. Show that for any X , the set $[X, I]$ has a single element.
- (b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

-
- (a) Follows from Exercise 51.3(a) and Exercise 51.3(c).
 - (b) Follows from Exercise 51.3(a) and Exercise 51.3(d).

EXERCISE 51.3

A space X is said to be contractible if the identity map $i_X : X \rightarrow X$ is nullhomotopic.

- (a) Show that I and $\mathbb{R}$ are nullhomotopic.
- (b) Show that a contractible space is path connected.
- (c) Show that if Y is contractible, then for any X , the set $[X, Y]$ has a single element.
- (d) Show that if X is contractible and Y path connected, then $[X, Y]$ has a single element.

- (a) Recall that nullhomotopic means being homotopic to a constant map. Let us denote by $c_0 : I \rightarrow I$ the constant map at 0. We have

$$H : I \times I \rightarrow I, \quad (x, t) \mapsto xt$$

which is a continuous map which satisfies

$$H(x, 0) = 0 = c_0(x) \quad \text{and} \quad H(x, 1) = x = i_I(x).$$

The corresponding map for $\mathbb{R}$ shows that also this space is contractible.

- (b) Let x and y be two points in the contractible space X ; we want to construct a path between them. Since X is contractible we can find a continuous map $H : X \times I \rightarrow X$ such that

$$H(w, 0) = \text{id}_X(w) = w \quad \text{and} \quad H(w, 1) = c_z(w) = z$$

where z is some point in X . Note that the restrictions

$$\alpha = H(x, t) : I \cong I \rightarrow X \quad \text{and} \quad \beta = H(y, t) : I \cong I \rightarrow X$$

are paths from x to z and from y to z , respectively. Then $\alpha * \bar{\beta}$ is a path from x to y , which shows that X is path connected.

- (c) Let $f : X \rightarrow Y$ be some continuous maps. Since Y is contractible, we know that we can find a continuous map $F : Y \times I \rightarrow Y$ such that

$$F(y, 0) = i_Y(y) = y \quad \text{and} \quad F(y, 1) = c_{y_0}(y) = y_0$$

where y_0 is some fixed point in Y . We want to show that f is homotopic to c_{y_0} . Consider map $G : X \times I \rightarrow Y$ defined as the composition

$$X \times I \xrightarrow{(f, \pi_2)} Y \times I \xrightarrow{F} Y, \quad .$$

This map is continuous, and it satisfies

$$G(x, 0) = H(f(x), 0) = f(x) \quad \text{and} \quad G(x, 1) = H(f(x), 1) = y_0 = c_{y_0}(x)$$

which shows that f is homotopic to the constant map c_{y_0} .

- (d) Let $f : X \rightarrow Y$ be a continuous map and let y_0 be some fixed point in Y . We want to show that f is homotopic to the constant map c_{y_0} . Since X is contractible we know that there is a continuous map $F : X \times I \rightarrow X$ such that

$$F(x, 0) = x \quad \text{and} \quad F(x, 1) = x_0$$

for some fixed x_0 in X . Since Y is path-connected we also know that there is a path between $f(x_0)$ and y_0 . That is, there is a continuous map $\gamma : I \rightarrow Y$ such that

$$\gamma(0) = f(x_0) \quad \text{and} \quad \gamma(1) = y_0.$$

We define a map $G : X \times I \rightarrow Y$ by

$$G(x, t) = \begin{cases} f(F(x, 2t)) & \text{if } 0 \leq t \leq 1/2 \\ \gamma(2t - 1) & \text{if } 1/2 \leq t \leq 1 \end{cases}.$$

Note that this is continuous by the gluing lemma and that

$$G(x, 0) = f(F(x, 0)) = f(x) \quad \text{and} \quad G(x, 1) = \gamma(1) = y_0$$

which shows that f is homotopic to c_{y_0} .

EXERCISE 52.4

Let $A \subset X$; suppose $r : X \rightarrow A$ is a continuous map such that $r(a) = a$ for each $a \in A$. If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \rightarrow \pi_1(A, a_0)$$

is surjective.

Note that $r : X \rightarrow A$ is a map such that $r \circ i = \text{id}_A$, where $i : A \rightarrow X$ denotes the inclusion. By Theorem 52.4 we have that

$$(r \circ i)_* = (\text{id}_A)_* \implies r_* \circ i_* = \text{id}_{\pi_1(A, a_0)}$$

which shows that r_* has a right inverse, and is hence surjective.

EXERCISE 53.1

Let Y have the discrete topology. Show that if $p : X \times Y \rightarrow X$ is the projection onto the first coordinate, then p covering map.

Let x be a point of X . Note that X is itself an open set containing x which is evenly covered by p . Indeed, we note that

$$p^{-1}(X) = X \times Y = \bigcup_{y \in Y} X \times \{y\}$$

where each slice $X \times \{y\}$ is open (since Y is discrete) and the restriction

$$p : X \times \{y\} \rightarrow X$$

is a homeomorphism.

EXERCISE 53.3

Let $p : E \rightarrow B$ be a covering map; let B be connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$. In such a case, E is called a k -fold covering of B .

Let B_k denote the subset of B consisting of points whose inverse image under p consists of k points. We know that this set consists of at least one point, namely b_0 . We claim that B_k is open in B . To prove this let $a \in B_k$. Since $p : E \rightarrow B$ is a covering map we know that a has a neighborhood U which is evenly covered by p . That is, we can write $p^{-1}(U)$ as a union of disjoint open sets

$$p^{-1}(U) = \bigcup_{i=1}^k V_i,$$

each mapping homeomorphically to U via p . Note that every point in U also belongs to B_k , so then U is an open neighborhood of a completely contained in B_k . Same argument to show that $B \setminus B_k$ is also open. Hence, B_k and $B \setminus B_k$ forms a separation of B , and since B is connected we conclude that $B \setminus B_k = \emptyset$.

EXERCISE 53.5

Show that the map of Example 3 is a covering map. Generalize to the map $p(z) = z^n$.

Let us use polar coordinates and write a point on the circle as $w = e^{i\theta}$. We claim that the open set

$$U = \left\{ e^{i\varphi} \mid \theta - \frac{\pi}{2} < \varphi < \theta + \frac{\pi}{2} \right\},$$

which contains w , is evenly covered by p . Indeed, note that the inverse image of a point on the circle is the two-point set

$$p^{-1}(e^{i\varphi}) = \left\{ e^{\frac{i\varphi}{2}}, e^{\frac{i\varphi}{2} + i\pi} \right\}$$

so that the inverse image of the set U is the two antipodal open sets “half the size of U ”. That we have a partition of the inverse image into two open sets that map homeomorphically to U :

$$p^{-1}(U) = \left\{ e^{i\varphi} \mid \frac{\theta}{2} - \frac{\pi}{4} < \varphi < \frac{\theta}{2} + \frac{\pi}{4} \right\} \cup \left\{ e^{i\varphi} \mid \frac{\theta}{2} + \frac{3\pi}{4} < \varphi < \frac{\theta}{2} + \frac{5\pi}{4} \right\}$$

In the general case $p(z) = z^n$, the inverse image of a point on the circle contains n points, so the inverse image of U consists of n -open sets “ $1/n$ the size of U ” centered as the n roots of w . That is, we have a partition of the inverse image into n sets that map homeomorphically to U :

$$p^{-1}(U) = \bigcup_{j=0}^{n-1} \left\{ e^{i\varphi} \mid \frac{\theta}{n} - \frac{\pi}{2n} + \frac{j\pi}{n} < \varphi < \frac{\theta}{n} + \frac{\pi}{2n} + \frac{j\pi}{n} \right\}.$$

Note that $p(z) = z^n$ is in particular an n -fold covering of S^1 .

PROBLEM 1

Show that the projection $S^n \rightarrow \mathbb{RP}^n$ is a covering map.

Recall that we can describe the real projective space as the quotient

$$\mathbb{RP}^n = S^n / \{\pm 1\}$$

where we have identified antipodal points on the sphere. In particular, we know that quotient maps associated to quotient spaces on this form are always open by Exercise session 7 Problem 4. Note also that the action of the group $\{\pm 1\}$ on the n -sphere is free; we have $gx \neq x$ whenever $g \neq 1$. Hence we can use the results in Exercise Session 7 Problem 5, as well. Part of that exercise was showing that every point in a quotient space obtained from quotienting out by a free action has a neighborhood U of x such that the G -translates were disjoint. That is, given $x \in S^n$ we can always find a neighborhood U of x such that the antipodal set

$$U_- = \{-x \in S^n \mid x \in U\}$$

is disjoint from U . Of course, then $p(U)$ is an open neighborhood of the equivalence class $[x] = \{x, -x\}$ in $\mathbb{RP}^n$, and it is evenly covered by p since

$$p^{-1}p(U) = U \cup U_- ,$$

both of which maps homeomorphically to $p(U)$ under p . This tells us that p is a 2-fold covering map.

Remark. Note that we are not using anything special about S^n and $\mathbb{RP}^n$, here. The same line of argument shows that if a finite group G acts freely on a space X , then the projection $X \rightarrow X/G$ is a $|G|$ -fold covering space.