

EXERCISE SESSION 4

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EXERCISE 22.2

- (a) Let $p : X \rightarrow Y$ be a continuous map. Show that if there is a continuous map $f : Y \rightarrow X$ such that $p \circ f$ equals the identity map of Y , then p is a quotient map.
- (b) If $A \subset X$, a retraction of X onto A is a continuous map $r : X \rightarrow A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.
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- (a) Recall that a surjective map $p : X \rightarrow Y$ is called a quotient map if a subset in U in Y is open if and only if $p^{-1}(U)$ is open in X . The existence of the right inverse f guarantees that p is surjective, and the fact that p is continuous implies that $p^{-1}(U)$ is open if U is open. To see that the converse also holds, let V be a set in Y such that $p^{-1}(V)$ is open in X . Since f is continuous we know that $f^{-1}(p^{-1}(V)) = (p \circ f)^{-1}(V) = V$ is open.
- (b) Let $i : A \rightarrow X$ be the inclusion of A in X ; this is a continuous map when A is given the subspace topology. By definition, a retraction is a continuous map such that $r \circ i$ is the identity on A , so it must be a quotient map by (a).

EXERCISE 22.3

Let $\pi_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the projection onto the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ given as

$$A = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \geq 0 \text{ or } y = 0\}.$$

Let $q : A \rightarrow \mathbb{R}$ be obtained by restricting π_1 . Show that q is a quotient map which is neither open nor closed.

To show that $q : A \rightarrow \mathbb{R}$ is a quotient map we can use the result of the previous exercise. Consider the map $f : \mathbb{R} \rightarrow A$ given by $f(x) = (x, 0)$. We have $(q \circ f)(x) = x$ for all $x \in \mathbb{R}$, and both f and q are continuous maps since they can be described by restricting the target or source of continuous maps, respectively.

To show that q is not open we can consider the open set $U = A \cap B((0, 3), 1)$ in A . The image of this set under q is $[0, 1]$ which is not open in $\mathbb{R}$.

To show that q is not closed we can consider the set $F = \{(x, x^{-1}) \mid x > 0\}$ which is closed in A , but whose image $(0, \infty)$ is not closed in $\mathbb{R}$.

EXERCISE 23.6

Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects $\text{Bd}(A)$.

(Closures are taken in X .) We recall that the boundary of A is defined as the intersection

$$\text{Bd}(A) = \overline{A} \cap \overline{X \setminus A}.$$

Note that $C \cap A$ and $C \cap (X \setminus A)$ are non-empty disjoint sets whose union is C . By Lemma 23.1 and the fact that C is connected we know that the two sets above cannot form a separation of C , so in particular we must have

$$\overline{C \cap A} \cap (C \cap (X \setminus A)) \neq \emptyset \quad \text{or} \quad (C \cap A) \cap \overline{C \cap (X \setminus A)} \neq \emptyset.$$

Since the closure of an intersection is contained in the intersection of the closures, this implies that

$$C \cap \overline{A} \cap (X \setminus A) \neq \emptyset \quad \text{or} \quad C \cap A \cap \overline{X \setminus A} \neq \emptyset.$$

Both of these are contained in $C \cap \overline{A} \cap \overline{X \setminus A}$, so we finally conclude that

$$C \cap \text{Bd}(A) \neq \emptyset.$$

EXERCISE 23.10

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of connected spaces; let X be the product space

$$X = \prod_{\alpha \in J} X_\alpha.$$

Let $\mathbf{a} = (a_\alpha)_{\alpha \in J}$ be a fixed point of X .

- (a) Given any finite subset K of J , let X_K denote the subspace of X consisting of all the points $\mathbf{x} = (x_\alpha)$ such that $x_\alpha = a_\alpha$ for all $\alpha \notin K$. Show that X_K is connected.
- (b) Show that the union Y of the spaces X_K is connected.
- (c) Show that X equals the closure of Y ; conclude that X is connected.

- (a) Note that X_K is the product space

$$X_K = \prod_{\alpha \in J} Y_\alpha, \quad Y_\alpha = \begin{cases} X_\alpha & \text{if } \alpha \in K \\ \{a_\alpha\} & \text{if } \alpha \notin K \end{cases},$$

so it is connected by Theorem 23.6 in [Mun00].

- (b) We note that $\mathbf{a} \in X_K$ for all K , so the union is connected by Theorem 23.3.
- (c) Let $\mathbf{x}$ be a point of X and consider a basis open neighborhood U of $\mathbf{x}$. By definition of the product topology we know that U is on the form

$$U = \prod_{\alpha \in J} U_\alpha$$

where $U_\alpha = X_\alpha$ for all but finitely many indices. Write K_0 for the set of α such that $U_\alpha \neq X_\alpha$. Note that we have $\mathbf{x} \in X_{K_0} \cap U$, so in particular U intersects the union of all the X_K . Hence $\mathbf{x}$ belongs to the closure of all of these.

EXERCISE 23.11

Let $p : X \rightarrow Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and if Y is connected, then X is connected.

Assume contradictory that X is not connected. Then we can find disjoint open sets U and V in X covering X . Since $p^{-1}(\{y\})$ is connected for every $y \in Y$, we know that $p^{-1}(\{y\})$ is contained entirely in either U or V . Let us construct the following disjoint sets in Y :

$$\begin{aligned} U' &= \{y \in Y \mid p^{-1}(\{y\}) \subset U\} \\ V' &= \{y \in Y \mid p^{-1}(\{y\}) \subset V\}. \end{aligned}$$

We claim that U' and V' are open sets. To show this we use the fact that p is a quotient map. Indeed, since p is a quotient map U' is open in Y if and only if $p^{-1}(U') = U$ is open, which is true by definition. Similarly for V' . Hence U' and V' form a separation of Y , which is a contradiction since Y was assumed to be connected.

EXERCISE 24.2

Let $f : S^1 \rightarrow \mathbb{R}$ be a continuous map. Show there exists a point $c \in S^1$ such that $f(c) = f(-c)$.

Consider the continuous function $g : S^1 \rightarrow \mathbb{R}$ defined by

$$g(x) = f(x) - f(-x).$$

Note that we in particular have $g(-x) = -g(x)$, so by the intermediate value theorem there is some $c \in S^1$ such that $g(c) = 0$, which is of course equivalent to $f(c) = f(-c)$.

REFERENCES

[Mun00] J. R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.