

EXERCISE SESSION 5

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TOPOLOGICAL GROUPS 3

Let H be a subspace of the topological group G . Show that if H is also a subgroup of G , then both H and $\overline{H}$ are topological groups.

Recall that a topological group G is a T_1 space with the structure of a group such that the multiplication $\mu : G \times G \rightarrow G$ and inversion $\iota : G \rightarrow G$ are continuous maps. We first note that H and $\overline{H}$ are also T_1 since they are subspaces of a T_1 -space. Since H is a subgroup of G we know that $\mu(H \times H) \subset H$ and $\iota(H) \subset H$ and we conclude that

$$\mu|_{H \times H} : H \times H \rightarrow H \quad \text{and} \quad \iota|_H : H \rightarrow H$$

are continuous maps by [Mun00, Theorem 18.2], since they are obtained by restricting the source and target of continuous maps. Similarly, since μ and ι are continuous, we have by [Mun00, Theorem 18.1] that

$$\mu(\overline{H} \times \overline{H}) = \mu(\overline{H \times H}) \subset \overline{\mu(H \times H)} \subset \overline{H}$$

and

$$\iota(\overline{H}) \subset \overline{\iota(H)} \subset \overline{H},$$

so that the restricted maps

$$\mu|_{\overline{H} \times \overline{H}} : \overline{H} \times \overline{H} \rightarrow \overline{H} \quad \text{and} \quad \iota|_{\overline{H}} : \overline{H} \rightarrow \overline{H}$$

are continuous by the same argument.

TOPOLOGICAL GROUPS 4

Let α be an element of G . Show that the maps $f_\alpha, g_\alpha : G \rightarrow G$ defined by

$$f_\alpha(x) = \alpha \cdot x \quad \text{and} \quad g_\alpha(x) = x \cdot \alpha$$

are homeomorphisms of G .

The map $f_\alpha : G \rightarrow G$ is continuous; it is the composition

$$G \xrightarrow{(c_\alpha, \text{id}_G)} G \times G \xrightarrow{\mu} G, \quad g \mapsto (\alpha, g) \mapsto \alpha \cdot x$$

and that both of these maps are continuous. It is straight forward to see that the inverse of f_α is $f_{\alpha^{-1}}$, which is again continuous by the same argument. Hence f_α is a homeomorphism. Same argument for g_α .

TOPOLOGICAL GROUPS 5

Let H be a subgroup of G . If $x \in G$, define $xH = \{x \cdot h \mid h \in H\}$; this set is called a left coset of H in G . Let G/H denote the collection of left cosets of H in G ; it is a partition of G . Give G/H the quotient topology.

- (a) Show that if $\alpha \in G$, the map f_α of the preceding exercise induces a homeomorphism of G/H carrying xH to $(\alpha \cdot x)H$.
- (b) Show that if H is a closed set in the topology of G , then one-point sets of G/H are closed.
- (c) Show that the quotient map $p : G \rightarrow G/H$ is open.
- (d) Show that if H is closed in the topology of G and is a normal subgroup of G , then G/H is a topological group.

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- (a) Let $h_\alpha = p \circ f_\alpha : G \rightarrow G/H$; it carries x to $(\alpha \cdot x)H$ and is a continuous map by the previous exercise. We will show that the maps in the diagram

$$\begin{array}{ccc} G & & \\ p \downarrow & \searrow h_\alpha & \\ G/H & \xrightarrow{\quad} & G/H \end{array}$$

satisfies the conditions of [Mun00, Theorem 22.2] so that we get an induced map in the diagram. We need to check that h_α is constant on each set $p^{-1}(\{yH\})$. Note that

$$p^{-1}(\{yH\}) = \{x \in G \mid xH = yH\}$$

so for all $x \in p^{-1}(\{yH\})$ we have

$$h_\alpha(x) = (\alpha \cdot x)H = \alpha(xH) = \alpha(yH) = (\alpha \cdot y)H.$$

We conclude that the induced map $\tilde{h}_\alpha : G/H \rightarrow G/H$ carrying xH to $(\alpha \cdot x)H$ is continuous. Same argument for $h_{\alpha^{-1}}$ shows that it is a homeomorphism.

- (b) We know that the quotient topology on G/H is such that

$$\{yH\} \text{ is closed in } G/H \iff p^{-1}(\{yH\}) \text{ is closed in } G,$$

so it suffices to show that the right hand side holds. Consider the continuous map $f_{y^{-1}} : G \rightarrow G$ and note that

$$\begin{aligned} f_{y^{-1}}^{-1}(H) &= \{x \in G \mid y^{-1} \cdot x \in H\} \\ &= \{x \in G \mid xH = yH\} \\ &= p^{-1}(\{yH\}) \end{aligned}$$

which is then closed since H is.

- (c) Let U be an open set of G ; we wish to show that $p(U)$ is open in G/H . Since p is a quotient map this is equivalent to showing that $p^{-1}(p(U))$ is open in G . Note that

$$\begin{aligned} p^{-1}(p(U)) &= \{x \cdot h \mid x \in U \text{ and } h \in H\} \\ &= \bigcup_{h \in H} g_h(U) \end{aligned}$$

which is open since each map $g_h : G \rightarrow G$ is a homeomorphism.

- (d) It is a well-known fact that G/H obtains a group structure if H is normal in G ; see your standard textbook on basic group theory. In particular, the multiplication and inversion are given in the obvious way

$$xH \cdot yH = (x \cdot y)H \quad \text{and} \quad (xH)^{-1} = x^{-1}H.$$

The previous exercise (b) shows that H being closed implies that G/H is T_1 . Left to show is that the maps $\mu : G/H \times G/H \rightarrow G/H$ and $\iota : G/H \rightarrow G/H$ are continuous. Let $m : G \times G \rightarrow G$ and $i : G \rightarrow G$ denote the continuous multiplication and inversion maps of G . Observe that ι is the unique map that fits into the diagram

$$\begin{array}{ccc} G & & \\ p \downarrow & \searrow p \circ i & \\ G/H & \xrightarrow{\quad \iota \quad} & G/H \end{array}$$

so it is continuous by [Mun00, Theorem 22.2] since the composition $p \circ i$ is constant on the sets $p^{-1}(\{yH\})$. Indeed, if $x \in p^{-1}(\{yH\})$ then

$$(p \circ i)(x) = p(x^{-1}) = x^{-1}H = (xH)^{-1} = (yH)^{-1}.$$

Similarly, μ is the unique map that fits into the diagram

$$\begin{array}{ccc} G \times G & & \\ p \times p \downarrow & \searrow (p \times p) \circ m & \\ G/H \times G/H & \xrightarrow{\mu} & G/H \end{array}$$

so again it is continuous by [Mun00, Theorem 22.2], since $p \circ m$ is constant on the sets $(p \times p)^{-1}(\{(y_1H, y_2H)\})$. Indeed, if $(x_1, x_2) \in p^{-1}(\{(y_1H, y_2H)\})$ then

$$(p \circ m)(x_1, x_2) = p(x_1 \cdot x_2) = (x_1 \cdot x_2)H = x_1H \cdot x_2H = y_1H \cdot y_2H.$$

Note that $p \times p : G \times G \rightarrow G/H \times G/H$ is not a priori a quotient map, which might be a problem, but since p is an open map by (c), and products of open maps are open, and surjective continuous open maps are quotient maps, we are fine.

EXERCISE 25.8

Let $p : X \rightarrow Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected.

Let U be an open set in Y and C a component of U . We wish to show that C is open in Y and since p is a quotient map this is equivalent to $p^{-1}(C)$ being open in X . Consider a point $x \in p^{-1}(C) \subset p^{-1}(U)$; we will show that we can find a neighborhood of x which is completely contained in $p^{-1}(C)$. In particular, we claim that that open set is the connected component of $p^{-1}(U)$ containing x . Let us call that guy D . We know that D is open since X is locally connected. We also know that $p(D)$ is connected by [Mun00, Theorem 23.5], so it must be contained entirely in some connected component of U ; in particular, this must be C since $p(x) \in C$. We conclude that

$$p(D) \subset C \implies D \subset p^{-1}(p(D)) \subset p^{-1}(C).$$

EXERCISE 25.9

Let G be a topological group; let C be the component of G containing the identity element e . Show that C is a normal subgroup of G .

We note that if $x \in G$, then the set $xC = f_x(C)$ is the connected component containing x (here $f_x : G \rightarrow G$ is the map that multiplies with x from the left; the same map we saw in earlier exercises). So, if x and y are both elements of C , so that $xC = C$ and $yC = C$, then $x \cdot y$ is also in C , since

$$(x \cdot y)C = x(yC) = xC = C.$$

Similarly, we have

$$C = (x^{-1} \cdot x)C = x^{-1}(xC) = x^{-1}C$$

so x^{-1} also belongs to C . We have shown that C is a subgroup of G . Left to show is that it is normal. This follows from the fact that $Cx = g_x(C)$ is also the connected component containing x , so that $xC = Cx$.

EXERCISE 26.3

Show that a finite union of compact subspaces of X is compact.

Let $\{A_i\}_{i=1}^n$ be a finite collection of compact subspaces of X and let $\{U_\alpha\}$ be an open cover of the union:

$$\bigcup_{i=1}^n A_i \subset \bigcup_{\alpha} U_\alpha.$$

Of course, $\{U_\alpha\}$ is a cover of A_i for all i . Since each A_i is compact we can find finite subcovers

$$A_i \subset U_{\alpha_{1,i}} \cup U_{\alpha_{2,i}} \cup \cdots \cup U_{\alpha_{m_i,i}}$$

Then it is clear that

$$\bigcup_{i=1}^n A_i \subset \bigcup_{i=1}^n \bigcup_{j=1}^{m_i} U_{\alpha_{j,i}}$$

so that the union of the compact spaces is compact, as well.

EXERCISE 26.4

Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Let X be the metric space, denote the associated metric by d , and let A be the compact subspace. Since metric spaces are Hausdorff it follows from [Mun00, Theorem 26.3] that A is closed. We now turn to showing that A is bounded. If A is empty, we are done, otherwise we let a be a point in A . The collection $\{B_d(a, n) \mid n \in \mathbb{N}\}$ is an open cover of A and since A is compact we know that

$$A \subset B_d(a, n_1) \cup \cdots \cup B_d(a, n_k)$$

for some finite subcover. Setting

$$n = \max\{n_1, \dots, n_k\}$$

we clearly have $A \subset B_d(a, n)$, so that A is bounded.

Consider $\mathbb{R}$ equipped with the bounded metric

$$\bar{d}(x, y) = \min\{d(x, y), 1\},$$

where d is the standard Euclidean metric on $\mathbb{R}$. The bounded metric $\bar{d}$ induces the same topology on $\mathbb{R}$ as d , so $\mathbb{R}$ is not compact, but the entire space $\mathbb{R}$ is closed and bounded in the bounded metric.

EXERCISE 26.7

Show that if Y is compact, then the projection $\pi_1 : X \times Y \rightarrow X$ is a closed map.

Let C be a closed set of $X \times Y$. We want to show that $\pi_1(C)$ is closed as well. This is of course equivalent to showing that $X \setminus \pi_1(C)$ is open. Pick a point $x \in X \setminus \pi_1(C)$. The slice $\{x\} \times Y$ is completely contained in the open set $N = (X \times Y) \setminus C$, so by the tube lemma we know that

$$W \times Y \subset N$$

where W is a neighborhood of x . In particular, W is completely contained in $X \setminus \pi_1(C)$.

EXERCISE 26.8

Let $f : X \rightarrow Y$; let Y be compact Hausdorff. Then f is continuous if and only if the graph of f ,

$$\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}$$

is closed.

We start by showing that f continuous implies that Γ_f is closed. Let us look at the map

$$\phi : X \times Y \rightarrow Y \times Y, \quad (x, y) \mapsto (f(x), y)$$

which is continuous since f is continuous. Let us denote the diagonal of Y by Δ , and note that

$$\phi^{-1}(\Delta) = \Gamma_f.$$

The Y being Hausdorff is equivalent to the diagonal Δ being closed, so in particular this tells us that Γ_f is closed, since it is the inverse image of a closed set under a continuous map.

We now show that Γ_f being closed implies that f is continuous. Let C be a closed set in Y ; we wish to show that $f^{-1}(C)$ is closed, as well. We first note that the set $(X \times C) \cap \Gamma_f$ is closed in $X \times Y$. In particular, we have

$$f^{-1}(C) = \pi_1((X \times C) \cap \Gamma_f)$$

and the right hand side is closed since π_1 is a closed map by the previous exercise.

REFERENCES

- [Mun00] J. R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.