

EXERCISE SESSION 6

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EXERCISE 26.11

Theorem 1. *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then*

$$Y = \bigcap_{A \in \mathcal{A}} A$$

is connected.

Proof. Let us write $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$. Since the collection $\mathcal{A}$ is simply ordered by proper inclusions it forms a nested sequence. By the discussion right after the proof of [Mun00, Theorem 26.9] the space Y is non-empty. It is also closed, and hence compact by [Mun00, Theorem 26.2]. To show that it is connected, assume contradictory that we can find a separation $Y = C \cup D$. Then C and D are closed; since they form a separation of Y they are both open and closed in Y , and since Y is closed in X it follows that they are also closed in X . We conclude that they are compact by [Mun00, Theorem 26.2]. We use the following lemma.

Lemma 2. *Let C and D be disjoint compact subspaces of the Hausdorff space X . Then there exist disjoint open sets U and V containing C and D , respectively.*

Proof. Exercise 26.5 from [Mun00]. □

With the lemma in mind let us consider the set

$$Z = \bigcap_{\alpha \in I} A_\alpha \setminus (U \cup V).$$

Note that each set $A_\alpha \setminus (U \cup V)$ is closed (use [Mun00, Exercise 17.4]). They are also non-empty; if $A \setminus (U \cup V)$ we would have $A \subset U \cup V$, so that U and V form a separation on A which would contradict the fact that A_α is connected. We also note that the collection $\{A_\alpha \setminus (U \cup V)\}_{\alpha \in I}$ is simply ordered by proper inclusions as well. It follows from the discussion right after [Mun00, Theorem 26.9] that the collection has the finite intersection property, with the conclusion that Z is non-empty. Say $y \in Z$. But then $y \in A \setminus (U \cup V)$ for all $A \in \mathcal{A}$, so in particular y is a point of Y which does not lie in $U \cup V$, and hence not in $C \cup D$. Contradiction. □

EXERCISE 27.2

Let X be a metric space with metric d ; let $A \subset X$ be nonempty.

- (a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.
- (b) Show that if A is compact, then $d(x, A) = d(x, a)$ for some $a \in A$.
- (c) Define the ϵ -neighborhood of A in X to be the set

$$U(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

- (d) Assume that A is compact; let U be an open set containing A . Show that some ϵ -neighborhood of A is contained in U .
- (e) Show the result in (d) need not hold if A is closed but not compact.

- (a) We recall that the distance from x to A is defined as

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

We know that $x \in \overline{A}$ if and only if the ϵ -ball $B_d(x, \epsilon)$ intersects A for all $\epsilon > 0$, which happens if and only if $d(x, A) = 0$.

- (b) Fixing $x \in X$ consider map $f : A \rightarrow \mathbb{R}$ defined as $f(b) = d(x, b)$. This map is obtained by restricting the continuous map $d : X \times X \rightarrow \mathbb{R}$, so it is continuous. By the extreme value theorem [Mun00, Theorem 27.4] we know that there is some point $a \in A$ such that $f(a) = d(x, a) \leq d(x, b) = f(b)$ for all $b \in A$. Hence

$$d(x, A) = d(x, a)$$

for this specific a .

- (c) We show that

$$U(A, \epsilon) = \bigcup_{a \in A} B_d(a, \epsilon)$$

by showing each one of the inclusions. If $x \in B_d(a, \epsilon)$ for some $a \in A$, then

$$d(x, A) < d(x, a) < \epsilon$$

which shows that $x \in U(A, \epsilon)$. For the other inclusion, let $y \in U(A, \epsilon)$. Then $\inf\{d(y, a) \mid a \in A\} < \epsilon$. It follows that there is some $a \in A$ such that $d(y, a) < \epsilon$, so that $y \in B_d(a, \epsilon)$, and consequently in the union of all the ϵ -balls.

- (d) Let a be a point of A . Since $a \in A \subset U$ and U is open we know that we can find $\epsilon_a > 0$ such that $B_d(a, \epsilon_a) \subset U$. The collection $\{B_d(a, \epsilon_a)\}_{a \in A}$ forms an open cover of A , and since A is compact we can find a finite subcover. Say

$$A \subset B(a_1, \epsilon_1) \cup \cdots \cup B(a_n, \epsilon_n).$$

Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then

$$A \subset U(A, \epsilon/2) \subset U$$

- (e) Take for example the graph $A = \{(x, 1/x) \mid x > 0\}$. This is closed in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$, but not compact, and no open set containing A will contain some ϵ -neighborhood of A .

EXERCISE 27.4

Show that a connected metric space having more than one point is uncountable.

Let us denote our connected metric space by X and its associated metric by d . Let x and y be two points in X . Let us look at the set

$$A = \text{im}(d(x, -)) = \{d(x, z) \in \mathbb{R} \mid z \in X\}.$$

We claim that the set $[0, d(x, y))$ is completely contained in A . If it was not, so that there was an $r \in [0, d(x, y))$ such that $r \notin A$ we could look at the two sets

$$U = \{z \in X \mid d(x, z) < r\} \quad \text{and} \quad V = \{z \in X \mid d(x, z) > r\}.$$

which are disjoint open sets whose union is X . This would contradict the connectedness of X . We now know that X contains at least as many elements as on the half-open interval $[0, d(x, y))$, so that it must be uncountable.

EXERCISE 27.5

Let X be a compact Hausdorff space; let $\{A_n\}$ be a countable collection of closed sets of X . Show that if each set has empty interior in X , then the union $\bigcup_{n=1}^{\infty} A_n$ has empty interior in X .

We first note that a set in X having empty interior is equivalent to the assertion that the complement is dense in X . So, what we wish to prove is that the closure of

$$X \setminus \bigcup_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} X \setminus A_n$$

is the entirety of X . We will use the following lemma.

Lemma 3. *Let X be compact Hausdorff. Let A be a closed set in X and let U be a neighborhood of a point x not lying in A . Then there is a neighborhood V of x such that $\bar{V} \cap A = \emptyset$ and $\bar{V} \subset U$.*

Proof. Since A a closed subset of a compact space, it is compact by [Mun00, Theorem 26.2]. We use [Mun00, Lemma 26.4] to conclude that there are disjoint opens W and Z containing x and A , respectively. Without loss of generality we can assume that W is contained in U ; if it is not, we take the intersection $U \cap W$ which is also a neighborhood of x . Furthermore, note that [Mun00, Lemma 26.4] shows that all compact Hausdorff spaces are regular in the sense of [Mun00, Section 31], so that we may find a neighborhood V of W such that $\bar{V} \subset W$ by [Mun00, Lemma 31.1]. $\square$

Let x be a point of X and consider an open set U containing x . Since $X \setminus A_1$ is dense in X by assumption we know that we can find $x_1 \in U \cap (X \setminus A_1)$. In other words, x_1 is a point of U not in A_1 . We employ Lemma 3 to conclude that there is a neighborhood U_1 of x_1 such that $\bar{U}_1 \cap A_1 = \emptyset$ and $\bar{U}_1 \subset U$.

Now we continue with the same procedure for U_1 ; since $X \setminus A_2$ is dense in X the intersection $U_1 \cap (X \setminus A_2)$ is non-empty, so we find a point x_2 in there, and then use Lemma 3 to find an neighborhood U_2 of x_2 such that $\bar{U}_2 \cap A_2 = \emptyset$ and $\bar{U}_2 \subset U_1$. Rise and repeat. By this procedure we get a nested sequence

$$\cdots \subset \bar{U}_n \subset \cdots \subset \bar{U}_2 \subset \bar{U}_1$$

of non-empty closed sets. By the discussion right after the proof of [Mun00, Theorem 26.9], we conclude that the intersection is non-empty:

$$\bigcap_{n=1}^{\infty} \bar{U}_n \neq \emptyset.$$

Let us say that y is a point in the intersection. Then y also belongs to U , since $\bar{U}_n \subset U$ for all n . On the other hand, since $\bar{U}_n \cap A_n = \emptyset$ so that $\bar{U}_n \subset X \setminus A_n$, we have

$$\bigcap_{n=1}^{\infty} \bar{U}_n \subset \bigcap_{n=1}^{\infty} X \setminus A_n.$$

It follows that y also belong to the right hand side, so that

$$y \in U \cap \bigcap_{n=1}^{\infty} X \setminus A_n,$$

which is what we wanted to prove.

Remark. Please compare the proof to [Mun00, Theorem 48.2].

EXERCISE 27.6

Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its “middle third” $(1/3, 2/3)$. Let A_2 be the set obtained from A_1 by deleting its “middle thirds” $(1/9, 2/9)$ and $(7/9, 8/9)$. In general, define A_n by the equation

$$A_n = A_{n-1} \setminus \left(\bigcup_{k=0}^{\infty} \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

The intersection

$$C = \bigcap_{n=0}^{\infty} A_n$$

is called the Cantor set; it is a subspace of $[0, 1]$.

- (a) Show that C is totally disconnected.
- (b) Show that C is compact.
- (c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C .
- (d) Show that C has no isolated points.
- (e) Conclude that C is uncountable.

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- (a) Recall that totally disconnected space is a space whose only connected subspaces are the one-point sets. By part (c) of this exercise we know that the space A_n is the union of 2^n closed intervals of length 3^{-n} . Let D be a subspace of C containing at least two distinct points. Let us call the two points x and y . We will show that D cannot be connected. Indeed, note that we can always find an n so that $|x - y| > 3^{-n}$. Then x and y must necessarily be in different closed intervals of A_n and we can pick r so that r is not in A_n and hence not in C . Then $D \cap [0, r)$ and $D \cap (r, 1]$ form a separation of D , so that D is not connected.
 - (b) Note that A_0 is closed, and A_n is constructed from A_{n-1} by removing something open, so each A_n is closed by induction on n . An intersection of closed sets is closed, so C is also closed. It is also evidently bounded, so C is compact.
 - (c) We show by induction on n that A_n is the union of 2^n disjoint closed intervals of length 3^{-n} . The base case $n = 0$ is clear; this is by definition the union of $1 = 2^0$ closed interval of length $1 = 3^{-0}$. Assume that the induction hypothesis is true for A_{n-1} . We recall that the way that A_n is constructed from A_{n-1} is by removing the “middle third” from all of the closed intervals. We conclude that every closed interval of A_{n-1} contributes two closed intervals in A_n of length $1/3$ of the original interval. In total, we get $2 \cdot 2^{n-1} = 2^n$ closed intervals of length $3^{-1} \cdot 3^{1-n} = 3^{-n}$.
 - (d) Recall that an isolated point is a point x so that we can find a neighborhood of x that contains no other points of C , so let us show that every neighborhood U of a point $x \in C$ contains a point other than x . Let us construct a sequence of closed intervals I_n for our chosen point x . We are going to let I_n be the unique closed interval of A_n containing x . Let us write $I_n = [a_n, b_n]$. We note that the two sequences have $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converges to x . By definition this means that there are integers N and M such that

$$a_n \in U, \quad n \geq N \quad \text{and} \quad b_m \in U, \quad m \geq M.$$

Actually, one of the sequences will eventually be x , since x is an element of C and hence the end point of some interval. Let us consider the other sequence. Since that sequence does not contain x , but converges to it, U will necessarily contain a point other than x . Hence x is not isolated.

- (e) This now follows directly from [Mun00, Theorem 27.7] since C is a non-empty compact Hausdorff space with no isolated points.

EXERCISE 29.1

Show that the rationals $\mathbb{Q}$ are not locally compact.

We proceed by proof by contradiction. Let q be a rational number, U a neighborhood of q , and suppose that we can find compact C such that $U \subset C$. Note that C is closed by [Mun00, Theorem 26.3] since $\mathbb{Q}$ is Hausdorff. In particular, since U is open we must have

$$(a, b) \cap \mathbb{Q} \subset U \subset C \implies [a, b] \cap \mathbb{Q} \subset \overline{C} = C,$$

for some $a < b$. It follows by [Mun00, Theorem 26.2] that $[a, b] \cap \mathbb{Q}$ is compact. This is a contradiction.

Why is this a contradiction? Well, turns out that we can show that $[a, b] \cap \mathbb{Q}$ is not compact. Indeed, let us show this.

Lemma 4. *The set $[a, b] \cap \mathbb{Q}$ is not compact in $\mathbb{Q}$.*

Proof. Since $[a, b]_{\mathbb{Q}} = [a, b] \cap \mathbb{Q}$ is a metrisable, we can use Theorem [Mun00, Theorem 28.2]. Let us show that $[a, b]_{\mathbb{Q}}$ is not sequentially compact by constructing a sequence with no convergent subsequence. To do so, pick an irrational number $r \in [a, b] \setminus \mathbb{Q}$. The set $(r - 1/n, r + 1/n)$ will intersect $[a, b]_{\mathbb{Q}}$ since $[a, b]_{\mathbb{Q}}$ is dense in $[a, b]$, so let us pick a rational number

$$q_n \in [a, b]_{\mathbb{Q}}.$$

The so constructed sequence $(q_n)_{n=1}^{\infty}$ is a sequence in $[a, b]_{\mathbb{Q}}$ with no convergent subsequence. Indeed, as a sequence in $[a, b]$ we would have $q_n \rightarrow r$ as $n \rightarrow \infty$, but r is not a rational number so the sequence does not converge in $[a, b]_{\mathbb{Q}}$. $\square$

EXERCISE 29.8

Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace $\{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

Let us write

$$K = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$$

and $\tilde{\mathbb{Z}}_+$ for the one-point compactification of $\mathbb{Z}_+$. We construct a map $\varphi : \tilde{\mathbb{Z}}_+ \rightarrow K$ by the formula

$$\varphi(n) = \begin{cases} 1/n & \text{if } n \in \mathbb{Z}_+ \\ 0 & \text{if } n = \infty. \end{cases}$$

Note that this is clearly a bijection. By [Mun00, Theorem 26.6] it suffices to show that φ is continuous to conclude that it is a homeomorphism, since $\tilde{\mathbb{Z}}_+$ is compact and K is Hausdorff. For reference let us recall what the open sets of $\tilde{\mathbb{Z}}_+$ look like. They are on two forms:

- (1) Open sets of $\mathbb{Z}_+$; so any subset of $\mathbb{Z}_+$, since $\mathbb{Z}_+$ is a discrete set.
- (2) Sets on the form $\tilde{\mathbb{Z}}_+ \setminus C$ where C is compact in $\mathbb{Z}_+$; note that the compact sets of $\mathbb{Z}_+$ are the finite sets, since $\mathbb{Z}_+$ is discrete.

Let us pick an open set of K ; we know that it can be written on the form $K \cap U$ where U is open in $\mathbb{R}$. We have two cases:

- (1) If $0 \notin U$, then the inverse image $\varphi^{-1}(K \cap U)$ is a set not containing ∞ , so it is an open set of $\tilde{\mathbb{Z}}_+$ on the first form above.
- (2) If $0 \in U$, then $K \cap U$ contains all but finitely many points in the set $\{1/n \mid n \in \mathbb{Z}_+\}$. We conclude that the inverse image is an open set on the second form above.

Hence φ is continuous, thus a homeomorphism.

REFERENCES

- [Mun00] J. R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000. Second edition.