

EXERCISE SESSION 7

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PROBLEM 1

Show that every compact metric space X has a countable dense subset.

For each positive integer n we consider the open cover of X defined as:

$$\mathcal{B}_n = \{B_d(x, 1/n) \mid x \in X\}.$$

Since X is compact we know that this can be refined to a finite cover, that is, that there is some finite set A_n such that $\{B_d(a, 1/n) \mid a \in A_n\}$ covers X . Set

$$A = \bigcup_{n=1}^{\infty} A_n;$$

this is a countable union of finite sets, hence countable. We claim that it is also dense in X . To show this, let $x \in X$ and consider some basis element $B_d(x, \epsilon)$ around x . There is some n such that $1/n \leq \epsilon$ and since

$$\{B_d(a, 1/n) \mid a \in A_n\}$$

covers X we conclude that we can find $a \in A_n \subset A$ such that $x \in B_d(a, 1/n)$. In particular, we then have

$$a \in A \cap B_d(x, 1/n) \subset A \cap B_d(x, \epsilon)$$

which shows that A is dense in X .

PROBLEM 2

Show that every metric space with a countable dense subset has a countable basis for its topology.

Let us call the metric space X , the corresponding metric by d , and the countable dense subset by A . We will show that the collection of balls

$$\mathcal{B} = \{B_d(a, 1/n) \mid a \in A, n \in \mathbb{N}\}$$

forms a countable basis for the topology on X . This is a countable set since we can write it as

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \{B_d(a, 1/n) \mid a \in A\},$$

so that it is clear that it is a countable union of countable sets. Left to show is that our collection forms a basis for the topology on X . We check the two conditions in the definition of a basis.

- (1) If $x \in X$, then denseness of A implies that the open ball $B_d(x, 1)$ intersects A , so that we may pick $a \in B_d(x, 1) \cap A$. Then $x \in B_d(a, 1)$.
- (2) Assume that $x \in B_d(a, 1/n) \cap B_d(b, 1/m)$. Since the intersection is open we can find positive integer k such that

$$B_d(x, 1/k) \subset B_d(a, 1/n) \cap B_d(b, 1/m).$$

Since A is dense in X we pick $c \in B_d(x, 1/2k) \cap A$. Then we have

$$B_d(c, 1/2k) \subset B_d(x, 1/k) \subset B_d(a, 1/n) \cap B_d(b, 1/m)$$

since if $y \in B_d(c, 1/2k)$ then by the triangle inequality

$$d(x, y) \leq d(x, c) + d(c, y) \leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.$$

PROBLEM 3

A map $f : X \rightarrow Y$ between topological spaces is called proper if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$. Now let $f : X \rightarrow Y$ be a continuous map between Hausdorff spaces. We extend f to a map $F : X^* \rightarrow Y^*$ by defining $F(\infty) = \infty$. Show that f is proper if and only if F is continuous.

We start by showing that f being proper implies that F is continuous. We recall that the open sets in the one-point compactification of X (and analogously of Y) are of two types:

- (1) Open sets of X .
- (2) Sets on the form $X^* \setminus K$ where K is compact in X .

We want to check that $F^{-1}(U)$ is open in X^* if U is open in Y^* . We divide up in the two cases depending on what sort of open set U is.

- (1) If U is an open set of type 1, then $F^{-1}(U) = f^{-1}(U)$ which is open since f is assumed to be continuous.
- (2) If U is an open set of type 2, so that $U = Y^* \setminus K$ for some compact K in Y , then we have

$$F^{-1}(Y^* \setminus K) = F^{-1}(Y^*) \setminus F^{-1}(K) = X^* \setminus f^{-1}(K).$$

Since f is proper we know that $f^{-1}(K)$ is compact in X , and we conclude that the inverse image is an open set of Type 2 in X .

We now prove that F being continuous implies that f is proper. Let K be a compact subset of Y . Then $Y^* \setminus K$ is open in the one-point compactification Y^* , so that continuousness of F implies that

$$F^{-1}(Y^* \setminus K) = X^* \setminus f^{-1}(K)$$

is open in X^* . But then $f^{-1}(K)$ is compact in X , so that f is a proper map.

PROBLEM 4

By a (left) action of a topological group G on a space X we mean a continuous map

$$\lambda : G \times X \rightarrow X, \quad (g, x) \mapsto gx$$

such that for all $g, h \in G$ one has

$$g(hx) = (gh)x, \quad ex = x$$

where $e \in G$ is the identity element. Given such an action we introduce an equivalence relation in X by declaring that $x \sim y$ if and only if there is a $g \in G$ such that $gx = y$. The equivalence classes are called the orbits of the action. The corresponding quotient space X/G is called the orbit space. Show that the projection map

$$p : X \rightarrow X/G$$

is an open map.

Let us write

$$\lambda_g : X \rightarrow X, \quad x \mapsto gx$$

for the map given by left multiplication of g on X . We note that this is a homeomorphism; the map is continuous since it can be written as the composition

$$X \xrightarrow{(c_g, \text{id})} G \times X \xrightarrow{\lambda} X$$

and it has the inverse $\lambda_{g^{-1}}$ which is continuous for the same reason. To introduce some notation, let us write

$$Gx = \{gx \mid g \in G\}$$

for the orbit of x . Given an open U in X we want to show that $p(U)$ is open in Y . This is of course equivalent to $p^{-1}(p(U))$ being open in X . We note that we can write

$$p^{-1}(p(U)) = \bigcup_{x \in U} Gx = \bigcup_{g \in G} \lambda_g(U).$$

The result now follows from the fact that λ_g is a homeomorphism for all $g \in G$, so that $\lambda_g(U)$ is open for all $g \in G$.

PROBLEM 5

In Problem 4, the action is called free if $gx \neq x$ whenever $g \neq e$. Show that if a finite group G acts freely on an n -manifold M then the orbit space M/G is also an n -manifold.

Recall that n -manifold is a Hausdorff, second countable space such that every point has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$. We check these three conditions in order.

Hausdorff: Let us consider the subset $\mathcal{O}$ in $M \times M$ defined as

$$\mathcal{O} = \{(x, gx) \in M \times M \mid x \in X, g \in G\}.$$

We note that $(x, y) \in \mathcal{O}$ if and only if x and y are in the same orbit. Let us start by showing that $\mathcal{O}$ is closed in $M \times M$, this will imply that M/G is Hausdorff. To prove that $\mathcal{O}$ is closed note that we have continuous maps

$$\lambda_g : M \rightarrow M, \quad x \mapsto gx,$$

for each $g \in G$. Let us denote their graphs by

$$\begin{aligned} \Gamma_g &= \{(x, \lambda_g(x)) \in M \times M \mid x \in X\} \\ &= \{(x, gx) \in M \times M \mid x \in X\}, \end{aligned}$$

and note that

$$\mathcal{O} = \bigcup_{g \in G} \Gamma_g.$$

That the graphs Γ_g are closed follows from the fact that M is Hausdorff. Indeed, this was one of the implications in Exercise 26.8. This implies that $\mathcal{O}$ is closed as well since it is then a finite union of closed sets.

We use the fact that $\mathcal{O}$ is closed to show that M/G is Hausdorff. Indeed, if $Gx \neq Gy$ are distinct points in M/G let us consider the point $(x, y) \in M \times M$. We know in particular that $(x, y) \notin \mathcal{O}$. Since $\mathcal{O}$ is closed we know that $(M \times M) \setminus \mathcal{O}$ is open, so we can find some basis element $U \times V$ in $M \times M$ containing (x, y) which does not intersect $\mathcal{O}$. Then $p(U)$ and $p(V)$ are disjoint open set containing Gx and Gy , respectively.

Second countable: It was shown during the lecture that if $p : X \rightarrow Y$ is a surjective, open, continuous map and $\mathcal{B}$ is a basis for X , then $p(\mathcal{B})$ is a basis for Y . We showed in Problem 4 that $p : M \rightarrow M/G$ is an open map. Since M is second countable, it follows that so is M/G .

Locally Euclidean: Some standard notation and terminology. For a fixed $x \in M$, we call the subgroup

$$G_x = \{g \in G \mid gx = x\}.$$

the stabiliser of x with respect to G . We show the following lemmas; the first lemma just uses that the group G is finite, the second one will use the freeness of the action.

Lemma 1. *For each $x \in M$, there is a neighborhood U of x such that*

$$U \cap gU \neq \emptyset \iff g \in G_x.$$

Proof. If $gx \neq x$, then by Hausdorffness of M we can find disjoint neighborhoods W_g and V_g of x and gx , respectively. Let us refine W_g a bit. Since $\lambda_g : M \rightarrow M$ is continuous for all $g \in G$, we can by [munkres, Theorem 18.1] find some neighborhood U_g of x such that $gU_g \subset V_g$. Without loss of generality we assume that $U_g \subset W_g$ (otherwise, we can just replace U_g with the intersection $U_g \cap W_g$). We define U to be the intersection of all these open sets:

$$U = \bigcap_{g \in G \setminus G_x} U_g.$$

This is an open set, since G is finite. □

Lemma 2. *If U is as above, then $p : U \rightarrow p(U)$ is a homeomorphism.*

Proof. Since the action is assumed to be free we have $G_x = \{e\}$ for all $x \in M$, so in particular we have that the U constructed in the previous lemma is such that $U \cap gU = \emptyset$ for all $g \neq e$. In this case, the inverse map

$$p^{-1} : p(U) \rightarrow U, \quad Gx \mapsto x$$

is well-defined, and is continuous since p is an open map. $\square$

We use these lemmas to prove that M/G is locally Euclidean. Let Gx be a point in M/G , and pick the neighborhood U of x as in the above lemmas. We also know that, since M is locally Euclidean, there is some neighborhood U_x of x and homeomorphism $\varphi_x : U_x \rightarrow V_x$ where V_x is an open subset of $\mathbb{R}^n$. We refine this homeomorphism by considering the restricted map

$$\varphi|_{U \cap U_x} : U \cap U_x \rightarrow \varphi(U \cap U_x)$$

which is then also a homeomorphism to an open subset of $\mathbb{R}^n$. However, the restricted neighborhood $U \cap U_x$ is better behaved than the neighborhood U_x alone, since $U \cap U_x$ is necessarily disjoint from its G -translates, so that

$$p : U \cap U_x \rightarrow p(U \cap U_x)$$

is a homeomorphism. By just composing these two homeomorphisms we have shown that the neighborhood $p(U \cap U_x)$ of Gx is homeomorphic to an open subset of $\mathbb{R}^n$, namely $\varphi(U \cap U_x)$.

REFERENCES