

EXERCISE SESSION 8

ALICE HEDENLUND

PROBLEM 1

The special unitary group $SU(n)$ consists of all $n \times n$ complex matrices A such that $A^*A = I$ and $\det A = 1$, where $A^* = \overline{A}^t$ is the conjugate transpose of A . Show that

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

Deduce from this that $SU(2)$ is homeomorphic to S^3 .

If we have $A^*A = I$, then we know that $A^* = A^{-1}$. Let us use the standard formula for the inverse of a 2×2 matrix

$$A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

We conclude that if the conjugate transpose of A is the inverse of A and if $\det(A) = 1$, then we must have

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

so that $d = \bar{a}$ and $c = -\bar{b}$. We conclude that an element of $SU(2)$ is on the form

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$$

with the restriction $|a|^2 + |b|^2 = 1$ to guarantee that the determinant is 1. To prove that $SU(2)$ is homeomorphic to S^3 , we use the description of S^3 as the unit sphere in $\mathbb{R}^4$:

$$S^3 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid |\mathbf{x}| = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

It is not hard to see that the map

$$\varphi : SU(2) \rightarrow S^3, \quad \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto (\Re(a), \Im(a), \Re(b), \Im(b))$$

is bijective with inverse

$$(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

and since $SU(2)$ is compact and S^3 is Hausdorff it suffices to show that it is continuous to guarantee that it is a homeomorphism. One way to see that φ is continuous is to remember how we topologise the matrix group $SU(2)$: a 2×2 complex matrix can be viewed as a vector in $\mathbb{R}^8$ by packing out the real and imaginary values of all the matrix entries

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \longleftrightarrow \quad (\Re(a), \Im(a), \Re(b), \Im(b), \Re(c), \Im(c), \Re(d), \Im(d))$$

so that we can view $SU(2)$ as a subspace of $\mathbb{R}^8$. With this point of view we note that the map φ is simply the restriction of the map

$$\psi : \mathbb{R}^8 \rightarrow \mathbb{R}^4, \quad (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_1, x_2, x_3, x_4)$$

which is clearly continuous since we only make use of the continuous projection maps. We conclude that φ is continuous since it is obtained by restricting the source and target of a continuous map.

PROBLEM 2

Show that the standard action of the topological group $\mathrm{GL}(n+1, \mathbb{C})$ on $\mathbb{C}^{n+1}$ induces an action of $\mathrm{GL}(n+1, \mathbb{C})$ on $\mathbb{CP}^n$.

Recall that complex projective space $\mathbb{CP}^n$ is defined as the quotient

$$\mathbb{CP}^n = \mathbb{C}^{n+1} / (\mathbf{v} \sim \lambda \mathbf{v}, \lambda \neq 0).$$

That is, we are identifying two non-zero vectors if they lie on the same line through the origin. Let us write the equivalence class of $\mathbf{v}$ as

$$[\mathbf{v}] = \{\lambda \mathbf{v} \mid 0 \neq \lambda \in \mathbb{C}\}.$$

The action of $A \in \mathrm{GL}(n+1, \mathbb{C})$ on $\mathbf{v} \in \mathbb{C}^{n+1}$ is simply obtained by multiplying the matrix with the vector:

$$A \cdot \mathbf{v} = A\mathbf{v}.$$

This induces the action

$$A \cdot [\mathbf{v}] = [A\mathbf{v}]$$

on complex projective space. We need to check that this is well-defined, or more specifically, that it is independent of the choice of representative of the class $[\mathbf{v}]$. Picking another representative $\lambda \mathbf{v}$ we have

$$A(\lambda \mathbf{v}) = \lambda(A\mathbf{v})$$

which is just another representative for the equivalence class of $[A\mathbf{v}]$, which is precisely what we wanted to show.

PROBLEM 3

Show that $\mathrm{GL}(2, \mathbb{C})$ acts on $\mathbb{CP}^1$ by Möbius transformations.

Recall that, in complex analysis, a Möbius transformation is a rational function

$$f(z) = \frac{az + b}{cz + d}$$

on one complex variable z , where a, b, c , and d are complex number such that $ac - bd \neq 0$. We can extend this to a self-map of the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ by setting

$$f(\infty) = \begin{cases} a/c & \text{if } c \neq 0 \\ \infty & \text{if } c = 0. \end{cases}$$

Note that we can identify the extended complex plane with the complex projective line via the map $\psi : \mathbb{C}^* \rightarrow \mathbb{CP}^1$ defined as

$$\psi(z) = \begin{cases} [z, 1] & \text{if } z \in \mathbb{C} \\ [1, 0] & \text{if } z = \infty. \end{cases}$$

Indeed, this map is bijective with inverse given by

$$\psi^{-1}([z, w]) = \begin{cases} z/w & \text{if } w \neq 0 \\ \infty & \text{if } w = 0. \end{cases}$$

Given an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathrm{GL}(\mathbb{C}, 2)$, let us now show that the composition

$$\mathbb{C}^* \xrightarrow{\psi} \mathbb{CP}^1 \xrightarrow{\lambda_A} \mathbb{CP}^1 \xrightarrow{\psi^{-1}} \mathbb{C}^*$$

is a Möbius transformation. Let us start by computing its value on some $z \in \mathbb{C}$:

$$\begin{aligned} \psi^{-1}(\lambda_A(\psi(z))) &= \psi^{-1}(\lambda_A([z, 1])) \\ &= \psi^{-1}([az + b : cz + d]) \\ &= \begin{cases} \frac{az+b}{cz+d} & \text{if } cz + d \neq 0 \\ \infty & \text{if } cz + d = 0. \end{cases} \end{aligned}$$

If $z = \infty$ we have

$$\begin{aligned} \psi^{-1}(\lambda_A(\psi(\infty))) &= \psi^{-1}(\lambda_A([1, 0])) \\ &= \psi^{-1}([a : c]) \\ &= \begin{cases} a/c & \text{if } c \neq 0 \\ \infty & \text{if } c = 0. \end{cases} \end{aligned}$$

which shows that $\mathrm{GL}(\mathbb{C}, 2)$ indeed acts on the complex projective line by Möbius transformations.

EXERCISE 31.1

Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

Let x and y be two distinct points of X . We start by finding disjoint open sets U_x and U_y containing x and y , respectively. We now use [Munkres, Lemma 31.1(a)] to conclude that we can find opens V_x and V_y containing x and y , respectively, such that $\overline{V_x} \subset U_x$ and $\overline{V_y} \subset U_y$. Since $U_x \cap U_y = \emptyset$, we conclude that $\overline{V_x} \cap \overline{V_y} = \emptyset$.

EXERCISE 31.5

Let $f, g : X \rightarrow Y$ be continuous; assume that Y is Hausdorff. Show that $\{x \in X \mid f(x) = g(x)\}$ is closed in X .

Let $\varphi : X \rightarrow Y \times Y$ be the map defined as $x \mapsto (f(x), g(x))$; it is continuous since each component is continuous. Recall that Y being Hausdorff is equivalent to the assertion that the diagonal $\Delta_Y = \{(y_1, y_2) \in Y \times Y \mid y_1 = y_2\}$ is closed in $Y \times Y$. We note that

$$\varphi^{-1}(\Delta_Y) = \{x \in X \mid f(x) = g(x)\}$$

which is then closed since it is the inverse image of a closed set.

EXERCISE 31.6

Let $p : X \rightarrow Y$ be a closed continuous surjective map. Show that if X is normal, then so is Y .

We will use the following lemma.

Lemma 1. *Let $p : X \rightarrow Y$ be a closed map. If A is a subset of Y and U is an open set containing $p^{-1}(A)$, then there is an open set W containing A such that $p^{-1}(W) \subset U$.*

Proof. We first show that if U is an open set containing $p^{-1}(\{y\})$, then there is a neighborhood W of y such that $p^{-1}(W) \subset U$. Indeed, if $p^{-1}(y) \subset U$, then

$$y \in Y \setminus p(X \setminus U),$$

which is an open set since $X \setminus U$ is closed and p is a closed map. Hence we find some neighborhood W of y completely contained in $Y \setminus p(X \setminus U)$. In other words, we find a neighborhood W of y such that

$$W \cap p(X \setminus U) = \emptyset.$$

This is equivalent to the statement that $p^{-1}(W) \subset U$. Indeed, the disjointment criterion means that

$$p(X \setminus U) \subset Y \setminus W \iff (x \notin U \Rightarrow p(x) \notin W).$$

The contrapositive of this statement is

$$(p(x) \in W \Rightarrow x \in U) \iff p^{-1}(W) \subset U.$$

For the general case of a subset A we use the above argument for each point $y \in A$ to find neighborhoods W_y such that $p^{-1}(W_y) \subset U$. It follows that the union

$$W = \bigcup_{y \in A} W_y$$

is an open set containing A with the wanted property. □

We now show that X being normal implies that Y is normal. First, we note that Y is T_1 : indeed, given a point y in Y we can write $p(x) = y$ for some $x \in X$ since p is surjective, and since X is T_1 and the map p closed it follows that $\{y\} = p(\{x\})$ is closed. Hence we conclude that $p^{-1}(\{y\})$ is closed for all y . Next, if A and B are disjoint closed sets in Y , then $p^{-1}(A)$ and $p^{-1}(B)$ are disjoint closed sets in X . Since X is normal we can separate them with disjoint open sets U_A and U_B . By the lemma above we can find opens W_A and W_B containing A and B , respectively, such that $p^{-1}(W_A) \subset U_A$ and $p^{-1}(W_B) \subset U_B$. Then W_A and W_B are necessarily disjoint as well which shows that Y is normal.