Exercise 2.2

2) Let $\mathcal{C}$ be the circle $|z - z_0| = r$. Let $f(z) = \frac{1}{z}$. We know that $f(\mathcal{C})$ is a $\bar{\mathbb{C}}$ circle so $f(\mathcal{C})$ is a straight line $\cup\, \infty$ iff $\infty \in f(\mathcal{C})$.

Now, $f(0) = \infty$ so $f(\mathcal{C})$ is a straight line $\cup\, \infty$ iff $0 \in \mathcal{C}$ iff $|z_0| = r$

3) $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$.

Assume first that $c \neq 0$

$$
\begin{array}{l}
az + b : cz + d = \frac{a}{c} \\
\underline{az + \frac{ad}{c}} \\
\qquad b - \frac{ad}{c}
\end{array}
$$

So $f(z) = \dfrac{a}{c} + \dfrac{b - \frac{ad}{c}}{cz + d} = (bc - ad)\dfrac{1}{c^2(z + \frac{d}{c})} + \dfrac{a}{c}$

This is clearly a composition of transformations of typ (i), (ii) and (iii).

If $c = 0$, then $f(z) = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)$

which is a composition of maps of type (i) and (ii).

2.2.3

To prove Lemma 2.2.1 it is enough to see that maps of type i), ii) and iii) map $\overline{\mathbb{C}}$ circles to $\overline{\mathbb{C}}$ circles.

For maps of type i) this is obvious.

Consider a circle $|z-z_0| = r$ in $\mathbb{C}$ and $f(z) = kz$. Put $w = kz$. Then $|w - kz_0| = |k|r$ so $f(C)$ is a circle with radius $|k|r$ and center kz_0.

Writing $z = x+iy$ a line l in $\mathbb{C}$ satisfy an equation of type

1) $ax + by = c$. Let $w = f(z) = kx + iky = u + iv$

We see that z satisfy 1 if and only if w satisfy 2) $au + bv = kc$ which again is the equation for another line l'. So $f(l) = l'$

Moreover $f(\infty) = \infty$ so $f(l \cup \infty) = l' \cup \infty$

Finally let $f(z) = \frac{1}{z}$

Consider the stereographic projection

$$\Phi : S^2 - \{(0,0,1)\} \longrightarrow \mathbb{C}$$

$$\Phi(x,y,z) = \frac{x}{1-z} + \frac{y}{1-z} i$$

with invers $\Phi^{-1}(u+iv) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$

This extends to a homeomorphism $\hat{\Phi} : S^2 \to \overline{\mathbb{C}}$ with $\hat{\Phi}((0,0,1)) = \infty$

Regarding $f: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ we get a map

$$\hat{\Phi}^{-1} \circ f \circ \hat{\Phi}(x,y,z) =$$

$$= \hat{\Phi}^{-1}\left(\frac{1}{\frac{x}{1-z} + \frac{y}{1-z}i}\right) = \hat{\Phi}^{-1}\left(\frac{(1-z)(x-yi)}{x^2+y^2}\right)$$

$$= \hat{\Phi}^{-1}\left(\frac{(1-z)(x-yi)}{1-z^2}\right) = \hat{\Phi}^{-1}\left(\frac{x-yi}{1+z}\right) =$$

$$= \cancel{\left(\frac{x-yi}{1+z}\right)} = \left(\frac{\frac{2x}{1+z}}{1+\frac{x^2}{(1+z)^2}+\frac{y^2}{(1+z)^2}}, \frac{\frac{-2y}{1+z}}{1+\frac{x^2}{(1+z)^2}+\frac{y^2}{(1+z)^2}}, \frac{\frac{x^2}{(1+z)^2}+\frac{y^2}{(1+z)^2}-1}{1+\frac{x^2}{(1+z)^2}+\frac{y^2}{(1+z)^2}}\right)$$

$$= \left(\frac{2x(1+z)}{(1+z)^2+x^2+y^2}, \frac{-2y(1+z)}{(1+z)^2+x^2+y^2}, \frac{x^2+y^2-(1+z)^2}{(1+z)^2+x^2+y^2}\right) =$$

$$= \left(\frac{2x(1+z)}{2+2z}, \frac{-2y(1+z)}{2+2z}, \frac{1-z^2-1-2z-z^2}{2+2z}\right) = \underline{(x,-y,-z)} = \hat{f}(x,y,z)$$

So $\hat{\Phi}^{-1} \circ f \circ \hat{\Phi}(x,y,z)$ is a an orthogonal map (a reflection through the origin in the y-z plane, with the x-axis fixed) and it therefore maps circles in S^2 to circles in S^2.

Since $\hat{\Phi}$ maps circles in S^2 to $\bar{\mathbb{C}}$-circles and vice versa, it is clear that $f = \hat{\Phi} \circ \hat{f} \circ \hat{\Phi}^{-1}$ maps $\bar{\mathbb{C}}$-circles to $\bar{\mathbb{C}}$-circles.

4) Assume $\dfrac{az+b}{cz+d} = \dfrac{a'z+b'}{c'z+d'}$

for each z. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$

corresponds to the F.L.T. $g(z)=z$

So assume $\overset{p(z)}{\cancel{g(z)}}\, z = \dfrac{ez+f}{gz+h}$ for all z

Then since $\cancel{g(\infty)=\infty}$ $p(\infty)=\infty$ $g=0$, since $p(0)=0$, f

So $p(z) = \dfrac{e}{h} z = kz$ with $k = \dfrac{e}{h}$

So any F.L.T ~~representing~~ which is equal the identity must be represented by kI where I is the identity matrix.

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = kI$

or $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

6) Let $f \in \text{Möb}(\mathbb{H})$. Let $\ell = \{iy \mid y \in \mathbb{R}\}$

Suppose $f(\ell) = \ell$

Assume first $f(z) = \dfrac{az+b}{cz+d}$ with $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$

Then (since $f(\bar{\mathbb{R}}) = \bar{\mathbb{R}}$) we must have $f(0)=0$ or $f(0)=\infty$

So either $b=0$ or $d=0$.

Assume $b=0$. Then $f(z) = \dfrac{az}{cz+d}$. Then we must have $f(\infty)=\infty$

so $c=0$ and $f(z) = \dfrac{a}{d}z$ with $ad=1$, $d=\frac{1}{a}$

so $f(z) = a^2 z$. for some $a>0$.

Assume $d=0$. Then $f(z) = \dfrac{az+b}{cz}$. Then $f(\infty)=0$

so $a=0$, so $f(z) = \dfrac{b}{cz}$. Now $bc=-1$, $c = -\frac{1}{b}$

and $f(z) = -\dfrac{b^2}{z}$ for some $b>0$.

Next assume $f(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$ with $ad-bc=-1$, $a,b,c,d \in \mathbb{R}$

Again we have either $f(0)=0$ and $f(\infty)=\infty$, so $b=c=0$

and $d=-\frac{1}{a}$ $\underline{f(z) = -a^2\bar{z}}$ for some $a>0$; or $f(0)=\infty$, $f(\infty)=0$ so $a=d=0$,

and $c=\frac{1}{b}$, so $f(z) = \dfrac{b^2}{\bar{z}}$ for some $b>0$

7) A group is acting transitively on a set if we only have one orbit.

a) Let (z_1, z_2, z_3), (w_1, w_2, w_3) be two tripples of distinct points in $\bar{\mathbb{R}}$. We know (by Corollary 2.2.5) that there exists a unique map $f \in \text{Möb}^+(\mathbb{C})$ with real coefficients s.t. $f(z_i) = w_i$. Then either

$f(\mathbb{H}) = \mathbb{H}$ and in this case $f \in \text{Möb}^+(\mathbb{H})$

or $f(\mathbb{H}) = \{iy \mid y < 0\}$

So assume $f(\mathbb{H}) = \{iy \mid y < 0\}$, $f(z) = \dfrac{az+b}{cz+d}$

In this case $ad - bc < 0$ and we may assume $ad - bc = -1$ (putting $a := \dfrac{a}{\sqrt{|ad-bc|}}$, etc.)

Let $g(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$, then since $z_i \in \bar{\mathbb{R}}$

$g(z_i) = f(z_i) = w_i$ and $g \in \text{Möb}^-(\mathbb{R}) \subset \text{Möb}(\mathbb{H})$.

Next let (ℓ_1, ℓ_2), (Γ_1, Γ_2) be two pairs of $\mathbb{H}$ lines with a common endpoint. Let the common endpoint of (ℓ_1, ℓ_2) be p and that of (Γ_1, Γ_2) be q.
Let the other end point of ℓ_i be p_i, $i = 1, 2$ and of Γ_i be q_i, $i = 1, 2$

Then we have proved that there exists $f \in \text{Möb}(\mathbb{H})$, such that $f(p) = q$ and $f(p_i) = q_i$, $i = 1, 2$

~~Hence f~~
Now an $\mathbb{H}$-line is uniquely determined by it endpoints and since f maps $\mathbb{H}$-lines to $\mathbb{H}$-lines, we must have $f(\ell_i) = \Gamma_i$, $i = 1, 2$.

b) It enough to show that given $\{p,q\} \in \overline{\mathbb{R}}$ $(p \neq q)$
~~Let $r \in \overline{\mathbb{R}}, r \neq p, r \neq q$.~~
then $\exists$ $f \in \text{Möb}^+(\mathbb{H})$ such that $f(p)=1$ and $f(q)=-1$
Let $r \in \overline{\mathbb{R}}, r \neq p, r \neq q$.
Now there exists ~~f~~ a unique f in $\text{Möb}^+(\mathbb{C})$

s.t. $f(p)=1$, $f(q)=-1$, $f(r)=0$, and $f(z) = \dfrac{az+b}{cz+d}$

with $a,b,c,d \in \mathbb{R}$. If $ad-bc>0$ then $f \in \text{Möb}^+(\mathbb{H})$

If $ad-bc<0$ consider $g(z) = \dfrac{cz+d}{az+b}$. Then $g \in \text{Möb}^+(\mathbb{H})$ $(cb-ad>0)$

and $g(p) = \dfrac{1}{f(p)} = 1$, $g(q) = \dfrac{1}{g(q)} = -1$.

c) Consider the points $(i, 2i)$ and $(i, 4i)$

Assume $\exists$ $h \in \text{Möb}(\mathbb{H})$ s.t. $h(i)=i$

$h(2i) = 4i$. Then ~~$h\{iy\}$~~

$h(\{iy \mid y \in \mathbb{R}\} \cup \{\infty\}) = \{iy \mid y \in \mathbb{R}\} \cup \{\infty\}$,

and we know from b) that either

$h(z) = a^2 z$ or $h(z) = -\dfrac{b^2}{z}$ or $h(z) = -a^2\overline{z}$

or $h(z) = \dfrac{b^2}{\overline{z}}$ with $a>0, b>0$. Now $h(i)=i \Rightarrow a^2=1$ or $b^2=1$

But $h(2i) \neq 4i$, when $h(z) = z, -\dfrac{1}{z}, -\overline{z}$ or $\dfrac{1}{\overline{z}}$