2.3.1

$$f(z) = \frac{4z-3}{2z-1} \quad (ad-bc = -4+6 = 2)$$

$$= \frac{\frac{4}{\sqrt{2}}z - \frac{3}{\sqrt{2}}}{\frac{2}{\sqrt{2}}z - \frac{1}{\sqrt{2}}} \qquad a = \frac{4}{\sqrt{2}},\ b = -\frac{3}{\sqrt{2}}$$

$$c = \frac{2}{\sqrt{2}},\ d = -\frac{1}{\sqrt{2}}$$

$(a+d)^2 = \frac{9}{2} > 4$, f is hyperbolic

Two real fixpoints

$$z = \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c} = \begin{cases} 1 \\ \frac{3}{2} \end{cases}$$

Let $h \in \text{Möb}^+(\mathbb{H})$ be s.t. $h(1) = 0$, $h(\frac{3}{2}) = \infty$

$$h(z) = -\frac{z-1}{z-\frac{3}{2}} = \frac{-2z+2}{2z-3} \in \text{Möb}^+(\mathbb{H})$$

$$h^{-1}(z) = \frac{-3z-2}{-2z-2}$$

So since $\begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

$$g(z) = h \circ f \circ h^{-1}(z) = \frac{4z}{2} = 2z = (\sqrt{2})^2 z$$

and $f = h^{-1} \circ g \circ h$

ii) $f(z) = \frac{-1}{z-1}$ $ad-bc = 0(-1)-(-1)\cdot 1 = 1$

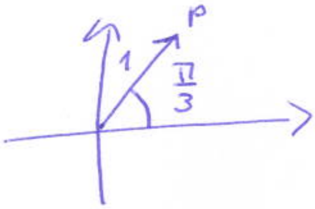
$a=0,\ b=-1,\ c=1,\ d=-1$

$(a+d)^2 = 1 < 4$

f is elliptic with two complex fix points

$f(z)=z \Longleftrightarrow z = \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$

Let $p = \frac{1}{2} + \frac{\sqrt{3}}{2} i$



$h(z) = \frac{z - Re(p)}{Im p} = \frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2z-1}{\sqrt{3}}$, then $h(p) = i$

$\begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} =$

$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \frac{4\theta}{3}$

So $g = h \circ f \circ h^{-1}$ corresponds to $\begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

with $\theta = \frac{\pi}{3}$, $g(z) = \frac{z+\sqrt{3}}{-\sqrt{3}z+1}$

and $f(z) = h^{-1} \circ g \circ h(z)$

2.3.1

c) $f(z) = \frac{4z-3}{2z-1}$ $(ad-bc = -4+6 = 2)$

$$= \frac{\frac{4}{\sqrt{2}}z - \frac{3}{\sqrt{2}}}{\frac{2}{\sqrt{2}}z - \frac{1}{\sqrt{2}}} \qquad a = \frac{4}{\sqrt{2}},\ b = -\frac{3}{\sqrt{2}}$$

$$c = \frac{2}{\sqrt{2}},\ d = -\frac{1}{\sqrt{2}}$$

$(a+d)^2 = \frac{9}{2} > 4$, f is hyperbolic

Two real fixpoints

$$z = \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c} = \begin{cases} 1 \\ \frac{3}{2} \end{cases}$$

Let $h \in \text{Möb}^+(\mathbb{H})$ be s.t. $h(1)=0,\ h(\frac{3}{2}) = \infty$

$$h(z) = -\frac{z-1}{z-\frac{3}{2}} = \frac{-2z+2}{2z-3} \in \text{Möb}^+(\mathbb{H})$$

$$h^{-1}(z) = \frac{-3z-2}{-2z-2}$$

So since $\begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

$$g(z) = h \circ f \circ h^{-1}(z) = \frac{4z}{2} = 2z = (\sqrt{2})^2 z$$

and $f = h^{-1} \circ g \circ h$

ii) $f(z) = \frac{-1}{z-1}$ $\quad$ $ad-bc = 0(-1)-(-1)\cdot 1 = 1$

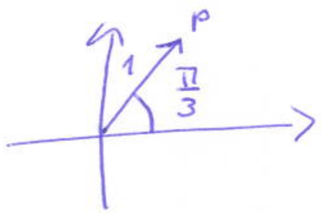
$a=0,\ b=-1,\ c=1,\ d=-1$

$(a+d)^2 = 1 < 4$

f is elliptic with two complex fix points

$$f(z)=z \iff z = \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

Let $p = \frac{1}{2} + \frac{\sqrt{3}}{2} i$



$$h(z) = \frac{z - \operatorname{Re}(p)}{\operatorname{Im} p} = \frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2z-1}{\sqrt{3}}$$, then $h(p) = i$

$$\begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{ with } \theta = \frac{4\pi}{3}$$

So $g = h \circ f \circ h^{-1}$ corresponds to $\begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

with $\theta = \frac{\pi}{3}$, $g(z) = \frac{z+\sqrt{3}}{-\sqrt{3}z+1}$

and $f(z) = h^{-1} \circ g \circ h(z)$

iii) $f(z) = \frac{z}{z+1}$ $a=1, b=0, c=1, d=1$

$ad-bc=1$

$(a+d)^2 = 4$, f is parabolic

with one real fixpoint equal $\frac{a-d}{2c} = 0$

Let $q=0, p=1, f(p)=\frac{1}{2}$

Consider $h(z) = [z, f(p), p, q]$

$= \frac{z-1}{z} \frac{(-1)}{} = \frac{1-z}{z}$, Note that here

$ad-bc = 0-1<0$ so $h \in Möb^-(\mathbb{C})$

and we must put $h := -h = \frac{z-1}{z}$.

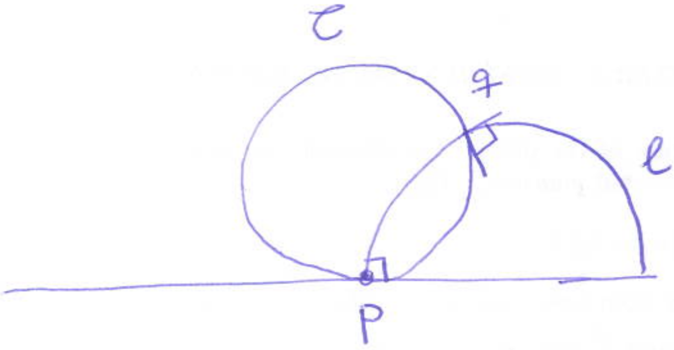
Now consider $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

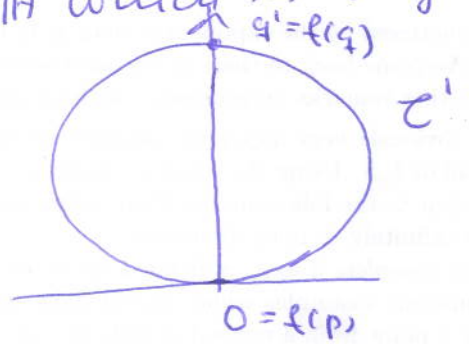
So $g = h \circ f \circ h^{-1} = z-1$

and $f = h^{-1} \circ g \circ h$

2.3.3



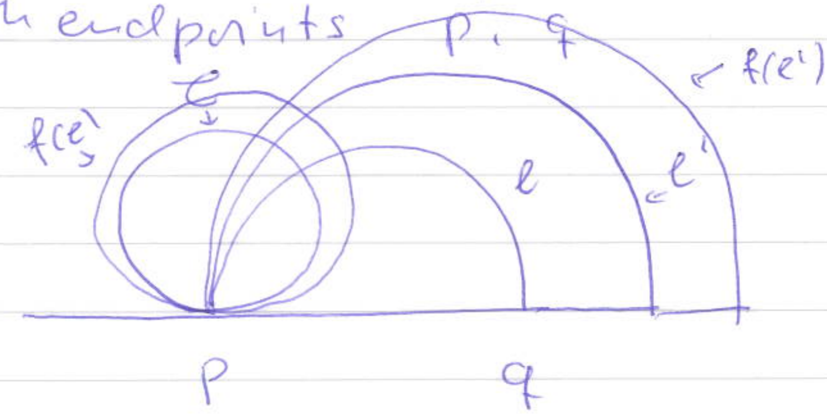
~~Sol~~ Let $f \in \text{Möb}^+(\mathbb{H})$, be such that $f(p) = 0$ and $f(\ell) = \{iy \mid y > 0\}$. Then $\mathcal{C}$ is mapped to a circle $\mathcal{C}'$ in $\mathbb{H}$ which is tangent to $\mathbb{R}$ at 0.



Such a circle must be symmetric around the imaginary axis. So 0 and q' is a diameter and the intersection at q' with the imaginary axis is orthogonal. Since f^{-1} preserves angles, the intersection between $\mathcal{C}$ and ℓ at q is also orthogonal. (That the intersection at p is orthogonal follows since ℓ intersects $\mathbb{R}$ at p orthogonally)

2.3

4) Let $f \in \text{Möb}^+(\mathbb{H})$, f hyperbolic with fixpoints p, q. Let ℓ be the $\mathbb{H}$-line with endpoints p, q



Let $\mathcal{C}$ be a horocircle at p.
Then $f(\mathcal{C})$ is a $\overline{\mathbb{C}}$ circle going through p (since $f(p)=p$)
and $f(\mathcal{C})-\{p\} \subset \mathbb{H}$. So $f(\mathcal{C})$ must be another horocircle at p.
Let ℓ' be another $\mathbb{H}$-line with end point p.
Then $f(\ell')$ is another $\mathbb{H}$-line, through p (since $f(p)=p$).

5) To show that all parabolic transformations are conjugate in $\text{Möb}(\mathbb{H})$ it is enough to show that $f(z) = z+1$ and $g(z) = z-1$ are conjugate in $\text{Möb}(\mathbb{H})$. Let $h(z) = -\bar{z}$, then $h^{-1} = h$.

So $h \circ f \circ h^{-1} = h \circ f \circ h = h(-\bar{z}+1) = -\overline{(-\bar{z}+1)} = -(-z+1) = z-1 = g(z)$

Let us show that $f(z)$ and $g(z)$ are <u>not</u> conjugate in $\text{Möb}^+(\mathbb{H})$. So, to obtain a contradiction assume $\exists\ h(z) = \frac{az+b}{cz+d}$ with $ad-bc=1$ s.t. $g = h \circ f \circ h^{-1}$. Since $h^{-1}(z) = \frac{dz-b}{-cz+a}$ we must have that

$$(*) \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = t \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ for some } t \neq 0.$$

Now the left hand side of the above equation $((*))$ is equal

$$\begin{bmatrix} 1-bc & a^2 \\ c^2 & 1+ac \end{bmatrix}$$

From $(*)$ follows that $c = 0$ so $t = 1$ but then $a^2 = -1$ which is impossible.

2.3.

6) Consider $g_\theta(z) = \dfrac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta}$, fix z, $z \neq i$

$\theta \in [0, \pi)$. If $\theta \neq \frac{\pi}{2}$, $\cos\theta \neq 0$ then

$g_\theta(z) = \dfrac{z + \tan\theta}{-\tan\theta\, z + 1} = \dfrac{z + w}{-wz + 1}$ with $w = \tan\theta$

So let $h_z(w) = \dfrac{z + w}{-wz + 1}$. Since $1 + z^2 \neq i$.

$h_z \in \text{Möb}^+(\mathbb{C})$ (for each z)

Note that when $w = \tan\theta$ and $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$

$\tan\theta \in \mathbb{R}$. So when θ varies and $\theta \in [0, \pi)$

w will trace through $\overline{\mathbb{R}}$. (∞, corresponds to $\theta = \frac{\pi}{2}$)

So the image of $\theta \to g_\theta(z)$ will be equal the image of $w \to h_z(w)$ when $w \in \overline{\mathbb{R}}$. Since F.L.T maps $\overline{\mathbb{C}}$ circles to $\overline{\mathbb{C}}$ circles this image will be a $\overline{\mathbb{C}}$ circle, since $h_z(\frac{1}{z}) = \infty$, and $\frac{1}{z} \notin \overline{\mathbb{R}}$ when $z \in \mathbb{H}$ ~~to~~ $\infty \notin h_z(\overline{\mathbb{R}})$ so $h_z(\overline{\mathbb{R}})$ is a circle in $\mathbb{C}$.

Problem 7

We know that a map $f \in \text{Möb}^-(\mathbb{H})$, $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, $ad-bc=-1$ is either an inversion (if $a+d=0$) then the fix pointset is an $\mathbb{H}$-line

(~~so the fixpoints~~

or if $a+d=0$ there are two fixpoints in $\overline{\mathbb{R}}$

So ~~the only~~ if we have fixpoints in $\mathbb{H}$ then f is an inversion

8) Let $f \in \text{Möb}^-(\mathbb{C})$ be an inversion in a circle C. We know that the fixpoints are the points in C, that f preserves angles and that the center z_0 of C is mapped to ∞ and vice versa and that the inside of C is mapped to the outside and vice versa. Also f maps $\overline{\mathbb{C}}$ circles to $\overline{\mathbb{C}}$ circles

(a) Let L be a straight line outside C.

Then $L \cup \infty$ is a $\overline{\mathbb{C}}$ circle which by f is mapped to a $\overline{\mathbb{C}}$ circle s.t. $f(\infty) = z_0$ (the center) and $f(L)$ is inside. But a $\overline{\mathbb{C}}$ circle inside C must be a circle in $\mathbb{C}$ an going, so $f(L \cup \infty)$ is a circle in inside C going through z_0.

b) Take a circle C' intersecting C orthogonally.

Then the intersection points are fixed.
so $f(\mathcal{C}')$ is a $\bar{\mathbb{C}}$ circle intersecting $\mathcal{C}$
in the same (two) points. Also $f(\mathcal{C}')$ intersects $\mathcal{C}$
orthogonally (since f preserves angles and $\mathcal{C}$ is fixed).
But since there is a unique circle intersecting
$\mathcal{C}$ in two given points $f(\mathcal{C}') = \mathcal{C}'$.

9) ~~Con~~

Since every element conjugate to an inversion is an inversion it is enough to show that any of the maps ① $z \pm 1$, ② $\lambda^2 z$, $\lambda \geq 1$ and $\frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$ ③ can be written as a composition of two inversions.

① Let $f(z) = z+1$, $g(z) = -\bar{z}$ and $h(z) = -\bar{z} - 1$

Then g, h are inversions and

$$g(h(z)) = g(-\bar{z} - 1) = -(\overline{-\bar{z} - 1}) = -(-z-1) = z+1 = f(z)$$

Similarly $h(g(z)) = h(-\bar{z}) = -(\overline{-\bar{z}}) - 1 = \underline{\underline{z-1}}$

Let $f(z) = \lambda^2 z$, $\lambda > 1$

② Let $g(z) = \frac{1}{\bar{z}}$, $h(z) = \frac{\lambda^2}{\bar{z}} = \frac{\lambda}{\frac{1}{\lambda}\bar{z}}$

Then g, h are inversions.

and $h(g(z)) = \frac{\lambda^2}{\overline{\frac{1}{\bar{z}}}} = \lambda^2 z$

③ Let $f(z) = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$, $\theta \in [0, \pi)$

If $\theta = 0$ then $f(z) = z = -(\overline{-\bar{z}})$ hence a composition of two inversions.

~~Assume $\theta \neq 0$. So $\cos\theta = \sin\theta \neq 0$~~

~~But~~ Let $g(z) = \frac{1}{\bar{z}}$ $h(z) = \frac{-\sin\theta \bar{z} + \cos\theta}{\cos\theta \bar{z} + \sin\theta}$. Then h and g are inversions

and $g(h(z)) = f(z)$.

10) It follows from

9) that $Möb^+(IH)$ are generated by inversions, and since compositions of two elements in $Möb^-(IH)$ is in $Möb^+(IH)$ and compositions of an element in $Möb^+(IH)$ and an element in $Möb^-(IH)$ are in $Möb^-(IH)$ it follows that any element in $Möb^+(IH)$ can be written only as a compositions of even number of inversions.

An element in $Möb^-(IH)$ is either an inversion or can be written as gh where g is an inversion and $h \in Möb^+(IH)$ so writting h as a composition of an even number of inversions we may write any element in $Möb^-(IH)$ as a composition of an odd number of inversions