Mandatory assignment in MAT 4510, fall 2014

Solutions should be handed in before 14.30, Thursday Oct. 30. For more information and rules regarding mandatory assignments, see link on the homepage.

Problem 1

(a) Let $g \in Möb^+(\mathbb{H})$ be defined by $g(z) = \dfrac{3z+4}{-z-1}$. Determine the type of g and write it explicitly as conjugate of a transformation on standard form.

(b) Assume that $f \in Möb^+(\mathbb{H})$ is not the identity, but that $f^n = f \circ \cdots \circ f$ (n times) is the identity for some integer n. Explain why f must be of elliptic type.

What can f be if it is on standard form?

(c) Given two arbitrary points z_1 and z_2 in $\mathbb{H}$. Show that we can find a parabolic transformation f and a hyperbolic transformation g such that $f(z_1) = z_2$ and $g(z_1) = z_2$.

(Hint: use the geometric descriptions of these two types of transformations.)

Show that there are exactly two possible choices for f and infinitely many choices for g.

Problem 2

(a) Calculate the hyperbolic arc–length of the (Euclidean) straight line segment from $3i$ to $4+5i$ in $\mathbb{H}$.

What is the hyperbolic distance between these two points?

(b) Calculate the hyperbolic area between the Euclidean segment in (a) and the hyperbolic segment between its endpoints.

(c) Find the equation for the hyperbolic circle with radius r and center i in $\mathbb{H}$.

Problem 3

(a) Consider the Poincare disk $\mathbb{D}$ as the interior of the unit disk in $\mathbb{C}$. Show that a Euclidean circle in $\mathbb{C}$ with center w and radius r intersects $\mathbb{D}$ in a $\mathbb{D}$–line if and only if $|w|^2 = r^2 + 1$. Explain why this means that there is a one–one correspondence between $\mathbb{D}$–lines not equal to a diameter and points w in $\mathbb{C}$ such that $|w| > 1$.

(b) Let w be such a point and let $\mathcal{L}$ be the corresponding $\mathbb{D}$–line. Find a formula for the inversion in $\mathcal{L}$.

Problem 4

Let the subgroup $G \subset M\ddot{o}b(\mathbb{C})$ be the intersection between the $M\ddot{o}b^+(\mathbb{H})$ and $M\ddot{o}b^+(\mathbb{D})$, considered as subgroups of $M\ddot{o}b(\mathbb{C})$. Show by calculation that as subgroup of each of the groups $M\ddot{o}b^+(\mathbb{H})$ and $M\ddot{o}b^+(\mathbb{D})$, G consists of hyperbolic transformations with the same axis.

How could we have seen this geometrically, without calculation?

Problem 6

(a) In a regular 4-gon the vertex angle is $\pi/3$. Find the lengths of its sides and diagonals.

(b) One particular such regular 4-gon in $\mathbb{H}$ has one vertex at i, one more in the ray $[i, \infty)$ and the last two vertices have positive real parts. Find all the vertices.

Problem 7

A compact, connected surface M has the presentation $D^2/ab^{-1}a^{-1}cdc^{-1}bd$. Find all possible pairs (m, n) such that M is homeomorphic to $S(m, n)$.

What is its Euler characteristic?

Is it orientable?