29th September, 2020

MAT4510 – Fall 2020

Mandatory assignment 1 of 1

Submission deadline

Thursday 15th October 2020, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Recall the definition of $\text{Möb}^-(\mathbb{H}), \text{Möb}^+(\mathbb{H}) \subset \text{Möb}(\mathbb{H})$.

a) Show that if $f_1, f_2 \in \text{Möb}^-(\mathbb{H})$, then $f_2 \circ f_1 \in \text{Möb}^+(\mathbb{H})$.

b) Let $z_1, z_2, w_1, w_2 \in \mathbb{H}$ and assume that $f \in \text{Möb}^-(\mathbb{H})$ satisfies $f(z_1) = w_1, f(z_2) = w_2$. Prove that there is a $g \in \text{Möb}^+(\mathbb{H})$ which satisfies $g(z_1) = w_1, g(z_2) = w_2$.

Problem 2. Write an introduction to elliptic geometry using the real projective plane as a model. Prove that Hilbert's axioms I1 – I3 are satisfied, but that B1 is not. Describe the set of axioms that is used instead of the betweenness axioms. What about parallel lines?

Problem 3. Recall the definition of $\text{Möb}^+(\mathbb{D}) \subset \text{Möb}(\mathbb{D})$.

a) Define a classification of the elements in $\text{Möb}^+(\mathbb{D})$ into parabolic, hyperbolic and elliptic elements, analogous to the classification of the elements of $\text{Möb}^+(\mathbb{H})$, by analyzing the fixpoints.

b) Show that $\frac{\sqrt{2}z+1}{z+\sqrt{2}} \in \text{Möb}^+(\mathbb{D})$ is hyperbolic, and find its axis.

c) Let l be the $\mathbb{D}$-line with endpoints i and 1, let m be the $\mathbb{D}$-line with endpoints -1 and i. Write down a parabolic Möbius transformation of $\mathbb{D}$ which maps l to m and which fixes i.

Problem 4. Recall the way we derived the line element and the area function

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \qquad A_{\mathbb{H}}(\Omega) = \iint_\Omega \frac{dx\,dy}{y^2}$$

in $\mathbb{H}$.

a) Derive formulas for the line element and area function in $\mathbb{D}$ by using that

$$\cosh(d_{\mathbb{D}}(z_1, z_2)) = 1 + \frac{2|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}.$$

<u>Hint:</u> You should get

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}, \qquad A_{\mathbb{D}}(\Omega) = \iint_\Omega \frac{4\,dx\,dy}{(1 - x^2 - y^2)^2}.$$

— Please turn —

b) Let $G\colon \mathbb{H} \to \mathbb{D}$, $G(z) = \frac{iz+1}{z+i}$ be our usual identification. Prove that $A_{\mathbb{D}}(G(\Omega)) = A_{\mathbb{H}}(\Omega)$.

c) Compute the (hyperbolic) area of the hyperbolic circular disk centered at $i \in \mathbb{H}$ of hyperbolic radius 1. Is it greater or smaller than the (euclidean) area of the euclidean disk of radius 1?