UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4510 — Goeometric Structures

Day of examination: Monday Dec 17. 2018.

Examination hours: 14.30 – 18.30

This problem set consists of 5 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All problems count for 10 points each. You have to explain all answers, and show enough details so that it is easy to follow your arguments. At the end of this document you will find some information that might be handy. You may answer the exam in either English or Norwegian.

Problem 1

Let $f \in \text{Möb}^+(\mathbb{H})$ be the map

$$f(z) = \frac{z-1}{z+1}.$$

Determine whether f is hyperbolic, parabolic or elliptic.

Solution:

To determine the type of $f(z)$ we have to decide whether f has a single fixed point in $\mathbb{H}$ (elliptic), a single fixed point on $\partial\mathbb{H}$ (parabolic), or two distinct fixed point on $\partial\mathbb{H}$ (hyperbolic). So we consider the equation $f(z) = z$, which is the same as

$$z - 1 = z(z+1) = z^2 + z,$$

which is the same as

$$z^2 = -1,$$

so f has the unique fixed point i in $\mathbb{H}$. Hence f is elliptic.

Problem 2

Let $S \subset \mathbb{R}^3$ be a smooth regular surface. Explain what the Gauss map $N : S \to S^2$ is, and use this to define Gaussian curvature.

Solution: Check the book.

Problem 3

Let $S \subset \mathbb{R}^3$ be a smooth regular surface, and let $\gamma : [0,1] \to S$ be a smooth curve. Explain geometrically what the covariant second derivative of γ is, and explain what it would mean for γ to be a geodesic.

Solution: Check the book.

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Problem 4

Let $g(u) = u, h(u) = e^{-u}$, and let

$$x(u, v) = (g(u)\cos(v), g(u)\sin(v), h(u)),$$

for $(u, v) \in (0, \infty) \times [0, 2\pi]$, parametrize the surface of revolution around the z-axis in $\mathbb{R}^3$ with coordinates (x, y, z).

(a) Compute the first fundamental form.

(b) Compute the Gaussian curvature.

(c) Prove that for a fixed v_0, the curves $x(u, v_0)$ are geodesics.

(d) Prove that for a fixed u_0, the curves $x(u_0, v)$ are not geodesics.

(e) Use the Gauss-Bonnet Theorem to prove that for any $0 < r < R < \infty$, the Euler characteristic of the "annulus" $x([r, R] \times [0, 2\pi])$ is zero.

Solution:
(a) We have that

$$x_u = (\cos(v), \sin(v), -e^{-u}),$$

and that

$$x_v = (-u\sin(v), u\cos(v), 0).$$

So we get that

$$E = x_u \cdot x_u = \cos^2(v) + \sin^2(v) + e^{-2u} = 1 + e^{-2u},$$

$$F = x_u \cdot x_v = \cos(v) \cdot (-u\sin(v)) + \sin(v) \cdot u\cos(v) + 0 = 0,$$

and

$$G = x_v \cdot x_v = u^2 \sin^v + u^2 \cos^2 v = u^2.$$

So the first fundamental form is

$$\mathbf{ds^2 = (1 + e^{-2u})du^2 + u^2 dv^2}.$$

(b) Here, the quickest is to remember the general formula for a surface of revolution

$$K(x(u, v)) = \frac{h'(u)(g'(u)h''(u) - h'(u)g''(u))}{g(u)(g'(u)^2 + h'(u)^2)^2}.$$

We have that $h'(u) = -e^{-u}, g(u) = u, g'(u) = 1, g''(u) = 0, h''(u) = e^{-u}$, and so

$$K(x(u, v)) = \frac{-e^{-u}(1 \cdot e^{-u} - 0)}{u(1 + e^{-2u})^2} = \frac{\mathbf{-e^{-2u}}}{\mathbf{u(1 + e^{-2u})^2}}.$$

Alternatively, one can calculate the coefficients of the second fundamental form e, f and g, and use the formula

$$K = \frac{ef - g^2}{EF - G^2}.$$

(Continued on page 3.)

(c) First we calculate the Christoffel symbols with respect to the parametrisation, using the fact at the end of the problem sheet. Here $E = 1 + e^{-2u}, F = 0$ and $G = u^2$, so we see that

$$\begin{bmatrix} 1+e^{-2u} & 0 \\ 0 & u^2 \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} -e^{-2u} & 0 & -u \\ 0 & u & 0 \end{bmatrix}$$

So we get that for a curve $\alpha(t) = (u(t), v(t))$ we have that

$$\begin{aligned} D\alpha''(t) &= (u''(t) + u'(t)^2\Gamma_{11}^1 + v'(t)^2\Gamma_{22}^1)x_u \\ &+ (v''(t) + u'(t)v'(t)\Gamma_{12}^2)x_v. \end{aligned}$$

For the curve $\alpha(t) = (t, v_0)$ we see that $v''(t) = v'(t) = 0$, so $D\alpha''(t)$ is indeed always a multiple of $\alpha'(t) = x_u$.

(d) If we consider the curve $\alpha(t) = (u_1, t)$ (which is even a constant speed curve), we see from (c) that the x_u-component of $D\alpha''(t)$ is $\Gamma_{22}^1 \cdot x_u$ which is never zero, so $D\alpha''(t)$ is never a multiple of $\alpha'(t) = x_v$.

(e) If we let α_r denote the curve $u = r$ and if we let α_R denote the curve $u = R$, the annular region $A \subset S$ is bounded by the smooth curves $x(\alpha_r)$ and $x(\alpha_R)$. We start by calculating the integral

$$\int_{\partial A} k_g ds,$$

and so first we parametrise α_r and α_R by arc length. We get $\alpha_r(t) = (r, t/r)$ and $\alpha_R(t) = (R, t/R)$. We get that

$$D\alpha_r''(t) = v'(t)^2\Gamma_{22}^1 x_u = (1/r)^2(-r)(1+e^{-2r})^{-1}x_u = \frac{-1}{r(1+e^{-2r})}x_u,$$

and similarly

$$D\alpha_R''(t) = \frac{-1}{R(1+e^{-2R})}x_u.$$

Now along α_r the inward pointing unit normal is $\frac{x_u}{\sqrt{1+e^{-2r}}}$ and so k_g^r becomes $\frac{-1}{r(1+e^{-2r})^{1/2}}$, and if we consider α_R the normal points the other way and k_g^R becomes $\frac{1}{R(1+e^{-2R})^{1/2}}$. It follows that

$$\int_{\partial A} k_g ds = 2\pi(\frac{1}{(1+e^{-2R})^{1/2}} - \frac{1}{(1+e^{-2r})^{1/2}}).$$

Next we consider the integral

$$\int\int_A K dA.$$

This is calculated in local coordinates

$$\begin{aligned} \int\int_A K dA &= \int_0^{2\pi}\int_r^R \frac{-e^{-2u}}{u(1+e^{-2u})^2}\sqrt{u^2(1+e^{-2u})}dudv \\ &= \int_0^{2\pi}\int_r^R \frac{-e^{-2u}}{(1+e^{-2u})^{3/2}}dudv \\ &= 2\pi[-(1+e^{-2u})^{-1/2}]_r^R \\ &= -\int_{\partial A} k_g ds. \end{aligned}$$

(Continued on page 4.)

The Gauss-Bonnet theorem states that

$$\int\int_A KdA + \int_{\partial A} k_g ds + \sum_k \epsilon_k = 2\pi\chi(A),$$

where the ϵ_k's are the turning angles, and $\chi(A)$ is the Euler characteristic. In this case ∂A is smooth, and so there are no turning angles, and it follows that $\chi(A) = 0$.

Problem 5

(a) State the classification theorem for compact connected surfaces.

(b) You may now take for granted that any compact connected surface admits a smooth structure, and a Riemannian metric with curvature constantly equal to -1, 0, or 1. For each of the surfaces in the classification in (a), determine the corresponding constant curvature.

Solution.
(a) Any compact connected surface is M homeomorphic to a surface

$$S(m,n) = \underbrace{T^2\#\cdots\#T^2}_{m}\#\underbrace{P^2\#\cdots\#P^2}_{n}$$

Furthermore, we have the relation $P^2\#P^2\#P^2 \approx T^2\#P^2$, which can be used to show that either

$$M \approx S(m,0) \tag{1}$$

or

$$M \approx S(0,n). \tag{2}$$

If M is orientable then M is of type (1) and m is uniquely determined by M (in fact $\chi(M)$), and if M is not orientable, then M is of type (2) and n is uniquely determined by M (in fact $\chi(M)$). (Here it is understood that $S(0,0) \approx S^2$). We have that $S(m,0)$ is never homeomorphic to $S(0,n)$ unless $m = n = 0$.

Alternatively, one has that $M \approx S(m,n)$ where $n = 0, 1$ or 2, and the pair (m,n) is uniquely determined by the orientability of M and $\chi(M)$.

(b) For a compact connected surface $M = S(m,n)$ we have that the Euler characteristic is given by

$$\chi(S(m,n)) = 2 - 2m - n.$$

In this case, the Gauss-Bonnet Theorem tells us that

$$\int\int_M KdA = 2\pi\chi(M),$$

where K denotes the Gaussian curvature. If $n = 0$ it follows that $K = 1$ if and only if $m = 0$, *i.e.*, if M is a sphere, it follows that $K = 0$ if and only if $m = 1$, *i.e.*, if M is a torus, and it follows that $K = -1$ if and only if $m \geq 2$. If $m = 0$ it follows that $K = 1$ if and only if $n = 0$ or $n = 1$, it follows that $K = 0$ if and only if $n = 2$, and it follows that $K = -1$ if and only if $n \geq 3$.

(Continued on page 5.)

THE END

Some facts:
Christoffel symbols for a metric $Edu^2 + 2Fdudv + Gdv^2$:

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \quad \begin{bmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{bmatrix} = \begin{bmatrix} E_u/2 & E_v/2 & F_v - G_u/2 \\ F_u - E_v/2 & G_u/2 & G_v/2 \end{bmatrix}$$

The covariant second derivative in local coordinates:

$$\begin{aligned} D\alpha''(t) = & \, (u''(t) + u'(t)^2\Gamma^1_{11} + 2u'(t)v'(t))\Gamma^1_{12} + v'(t)^2\Gamma^1_{22})x_u \\ & + (v''(t) + u'(t)^2\Gamma^2_{11} + 2u'(t)v'(t))\Gamma^2_{12} + v'(t)^2\Gamma^2_{22})x_v \end{aligned}$$