UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4510 — Geometric Structures

Day of examination: Monday Dec 02. 2019.

Examination hours: 09:00 – 13:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All problems, 1 a),1b) etc., count for 10 points each. You have to explain all answers, and show enough details so that it is easy to follow your arguments. At the end of this document you will find some information that might be handy. You may answer the exam in either English or Norwegian.

Problem 1

In this problem we consider the surface of revolution S parametrised by the map $x : (0, \infty) \times (-\infty, \infty) \to \mathbb{R}^3$ given by

$$x(u, v) = (u \cos(v), u \sin(v), \ln(u)).$$

(a) Compute the first fundamental form.

(b) Compute the Gauss map on S with respect to the parametrisation x.

(c) Define the Gaussian curvature in terms of the Gauss map.

(d) Compute the Gaussian curvature.

(e) Letting $\gamma(t) = (t, v_0)$ for a fixed v_0, prove that $x(\gamma)$ is a geodesic. Is $x(\gamma)$ a constant speed geodesic?

Problem 2

(a) Explain how a torus is constructed as a quotient of a square. Explain how a general closed compact surface is constructed as a quotient of an n-gon.

(b) State the topological classification of all closed compact surfaces. With respect to this classification, which surface is the following

$$X = D^2 \setminus bc^{-1}ab^{-1}c^{-1}a?$$

Is X orientable?

Problem 3

(Continued on page 2.)

(a) Elements of $\text{Mob}^+(\mathbb{H})$ are classified in three distinct categories. Define each of these categories.

(b) Classify the element $f(z) = \frac{2z}{-\frac{3}{2}z+\frac{1}{2}}$.

(c) Prove that g is hyperbolic if and only if g may we written

$$g(z) = \frac{(\lambda ad - \frac{bc}{\lambda})z + (\lambda bd - \frac{bd}{\lambda})}{(-\lambda ac + \frac{ac}{\lambda})z + (-\lambda bc + \frac{ad}{\lambda})}$$

where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ and $\lambda > 0$, $\lambda \neq 1$.

THE END

Some facts:

(1) Christoffel symbols for a metric $Edu^2 + 2Fdudv + Gdv^2$:

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} E_u/2 & E_v/2 & F_v - G_u/2 \\ F_u - E_v/2 & G_u/2 & G_v/2 \end{bmatrix}$$

The covariant second derivative in local coordinates:

$$\begin{aligned} D\alpha''(t) &= (u''(t) + u'(t)^2\Gamma_{11}^1 + 2u'(t)v'(t))\Gamma_{12}^1 + v'(t)^2\Gamma_{22}^1)x_u \\ &\quad + (v''(t) + u'(t)^2\Gamma_{11}^2 + 2u'(t)v'(t))\Gamma_{12}^2 + v'(t)^2\Gamma_{22}^2)x_v \end{aligned}$$

(2) The function $g : (0, \infty) \to \mathbb{R}$ defined by $g(t) = t + 1/t$, satisfies $g(t) > 2$ for all $t \in (0, \infty) \setminus \{1\}$.