UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT4510 — Geometrical structures.

Day of examination: Tuesday, December 13, 2011.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each item (1, 2, 4a, 4b etc.) counts 10 points. The integral formulae $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$ may be useful.

Problem 1

Let $f \in \text{Möb}^+(\mathbb{H})$ be defined by

$$f(z) = \frac{\sqrt{3}z - 1}{z}.$$

Decide wether f is parabolic, hyperbolic or elliptic. Write f explicitly as a conjugate of a map on normal form.

Problem 2

Consider a hyperbolic triangle with angles α, β and γ and with opposite sides of hyperbolic length a, b and c respectively. Formulate the first hyperbolic law of cosines and the hyperbolic law of sines for such triangle.
Use these laws together with the second law of cosines,

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a,$$

to prove that for a right triangle with say, $\gamma = \frac{\pi}{2}$, then $\cos\beta = \frac{\tanh a}{\tanh c}$.

Problem 3

Let R be the bounded region in $\mathbb{H}$ bounded by the circles $|z| = 1$, $|z| = \sqrt{2}$ and the lines $x = \pm\frac{1}{2}\sqrt{2}$. Calculate the hyperbolic area of R.

Problem 4

a) A regular surface $S \subset \mathbb{R}^3$ is parametrized by

$$\mathbf{x}(u, v) = (v\cos u, v\sin u, \frac{u^2}{2}),\ u, v > 0.$$

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Find the first fundamental form of S with respect to this parametrization and find the Gaussian curvature of S.

b) Another regular surface $S' \subset \mathbb{R}^3$ is parametrized by

$$\mathbf{y}(u, v) = (v \cos u, v \sin u, v),\ u \in (0, 2\pi), v > 0.$$

Let $f : S' \to \mathbb{R}^2$ be given by

$$f(\mathbf{y}(u, v)) = (\sqrt{2}v \cos \frac{u}{\sqrt{2}}, \sqrt{2}v \sin \frac{u}{\sqrt{2}}).$$

f is a diffeomorphism of S' onto an open set $U \subset \mathbb{R}^2$ (you are not supposed to prove this). Show that f is an isometry from S' to U. Does there exist any local isometry between open sets in S' and in the surface S in a) ?

c) Show that the curves on S' given by $u = \textit{constant}$ are geodesics. Let a, b and c be constants with $(a, b) \neq (0, 0)$. Show that curves $(u(t), v(t))$ with $(u'(t), v'(t)) \neq (0, 0)$ such that $av(t) \cos \frac{u(t)}{\sqrt{2}} + bv(t) \sin \frac{u(t)}{\sqrt{2}} = c$ correspond to geodesics on S'.

Problem 5

Let $\mathbb{H}$ have hyperbolic Riemannian metric. Let α be the line in $\mathbb{H}$ parametrized by $\alpha(t) = t + i(t + 1), t > -1$. Let R be the bounded region in $\mathbb{H}$, contained in $\{z : Im\, z \geq 1\}$, bounded by α, the circle $|z + 1| = 2$ and the imaginary axis.

a) Calculate the hyperbolic area of R by a direct integration.

b) Consider the portion of α where $t \in [0, t_0]$, $t_0 > 0$. Calculate the hyperbolic arc-length of this portion of α, and find the parametrization of α by arc-length starting at the point i.

c) In $\mathbb{H}$, the covariant second derivative for a curve $\beta(s) = x(s) + iy(s)$ parametrized by arc-length, is given by

$$D\beta'' = (x'' - \frac{2}{y}x'y') + i(y'' + \frac{1}{y}(x')^2 - \frac{1}{y}(y')^2).$$

Calculate the geodesic curvature $k_g(s)$ for the curve α given above. Here, we let the unit normal vector $n_\alpha(s)$ have positive y-component. Let α_1 be the portion of α with end-points i and $(\sqrt{2} - 1) + i\sqrt{2}$. Calculate $\int_{\alpha_1} k_g ds$.

d) Verify the Gauss-Bonnet theorem for the region R given above.

THE END