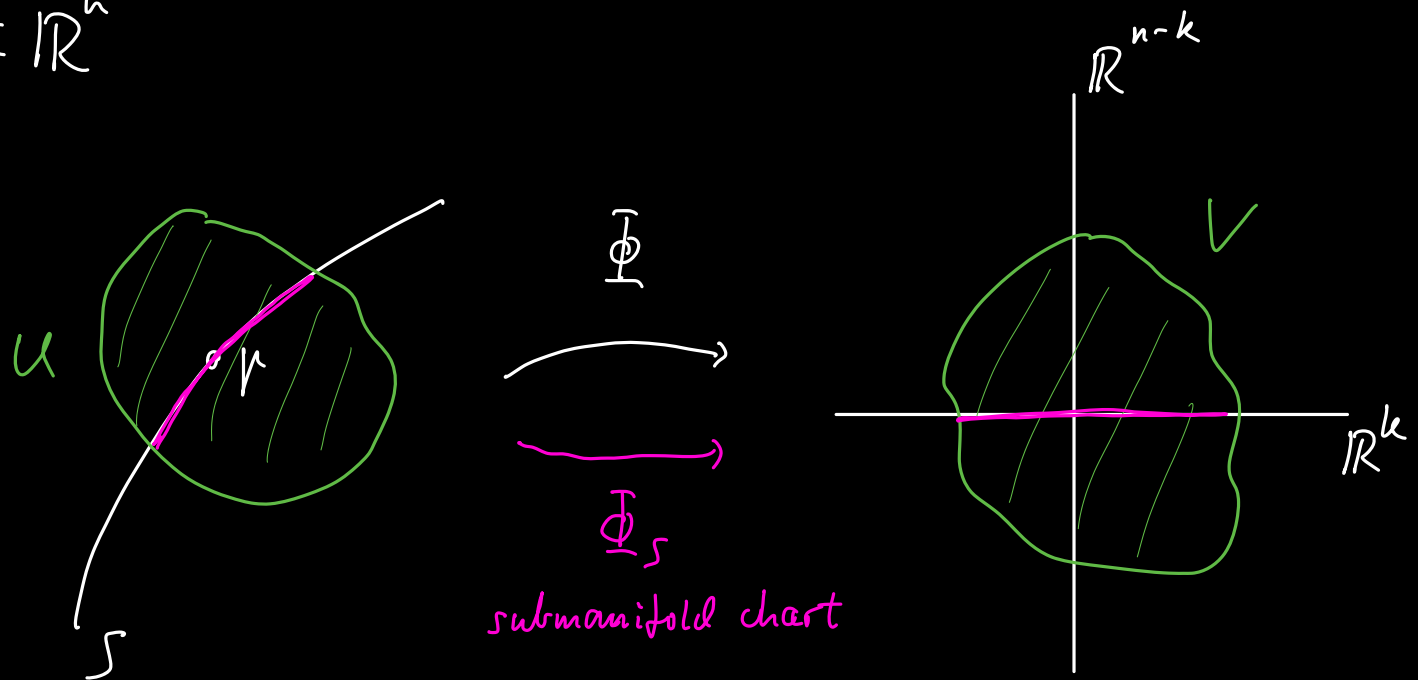


Submanifolds of $\mathbb{R}^n$ (cont'd)

Recall the definition of a submfd:

$$S \subset \mathbb{R}^n$$



$$\Phi(S \cap U) = (\mathbb{R}^k \times [0,1]) \cap V$$

Example: Any k -dim. linear subspace

$W \subset \mathbb{R}^n$ is a k -dim. submfd:

Proof: There is a linear isomorphism

$\Phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\Phi(W) = \mathbb{R}^n \times \{0\}. \quad //$$

Def: Let $\Omega \subset \mathbb{R}^n$ ^{open} and $F: \Omega \rightarrow \mathbb{R}^m$ smooth.

(i) A point $x \in \Omega$ is called a regular point of F if $dF_x: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is surjective; otherwise, x is called a singular point.

(ii) A point $y \in \mathbb{R}^m$ is called a regular value of F if every $x \in F^{-1}(y)$ is a regular point; otherwise, y is called a singular value.

Sard's Thm: The set of singular values has measure 0, hence the set of regular values is dense in $\mathbb{R}^m$.

Thm: If $\Omega \subset^{\text{open}} \mathbb{R}^n$ and c is a regular value of $F: \Omega \rightarrow \mathbb{R}^m$ then $S := F^{-1}(c)$ is a submanifold of $\mathbb{R}^n$ of dim. $n-m$.

Proof: After replacing F by $F - c$ we may assume that $c = 0$. Let $p \in S$ and $K := \ker(dF_p)$. Then $\dim K = n-m$.
Choose a linear map $A: \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$ which restricts to a linear isomorphism

$K \rightarrow \mathbb{R}^{n-m}$. Defn

$$G: \Omega \rightarrow \mathbb{R}^{n-m} \times \mathbb{R}^m = \mathbb{R}^n$$

$$x \mapsto (Ax, F(x)).$$

$$dG_x(v) = (Av, dF_x(v))$$

Then $dG_x: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is injective,

hence dG_x is an isomorphism.

$\Rightarrow$ G maps some nbhd $U \subset \Omega$ of x
inv. fn. thm.

diffomorphically onto an open set

$V \subset \mathbb{R}^{n-m} \times \mathbb{R}^m$. Then

$$G(S \cap U) = (\mathbb{R}^{n-m} \times \{0\}) \cap V. //$$

Example: ① $F: \mathbb{R}^n \rightarrow \mathbb{R}$, $F(x) = \|x\|^2$

$$d\bar{F}_x(v) = \langle \nabla \bar{F}_x, v \rangle,$$

$$\nabla \bar{F}_x = 2x \quad \Rightarrow \quad 0 \in \mathbb{R} \text{ is the only singular value}$$

$$\Rightarrow S^{n-1} = \bar{F}^{-1}(1) \text{ is an } (n-1)\text{-dim.}$$

submfd of $\mathbb{R}^n$.

$$= \mathbb{R}^{n^2}$$

$$(2) \quad \bar{F} = \det : \overbrace{M_{n \times n}(\mathbb{R})}^{= \mathbb{R}^{n^2}} \rightarrow \mathbb{R}$$

$$\det(tA) = t^n \det(A)$$

$$\frac{d}{dt} \det(tA) = n t^{n-1} \det(A)$$

$$d\bar{F}_A(A) = \left. \frac{d}{dt} \right|_{t=1} \det(tA) = n \det(A)$$

$\Rightarrow$ If $\det(A) \neq 0$ then A is a regular point of $\bar{F}$.

$$\Rightarrow SL_n(\mathbb{R}) = \bar{F}^{-1}(1) \text{ is a } (n^2-1)\text{-dim.}$$

subfld of $M_{n \times n}(\mathbb{R})$.

③ If $X \subset \mathbb{R}^m$ k -dim subfld
 $Y \subset \mathbb{R}^n$ l -dim.

$\Rightarrow X \times Y$ is a $(k+l)$ -dim. subfld
of $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

Example: $S^1 \times S^1 \subset \mathbb{R}^4$ torus

$$S^m \times S^n \subset \mathbb{R}^{m+n+2}$$

④ Any open subset of a subfld of $\mathbb{R}^n$
is again a subfld.

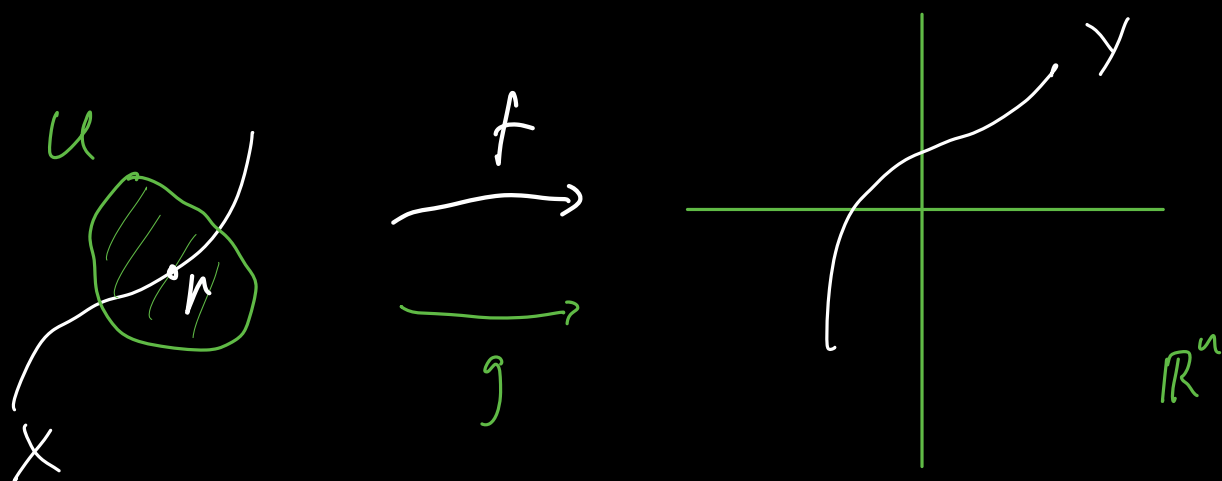
Def: Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be subflds.

A map $f: X \rightarrow Y$ is called smooth

if for every $p \in X$ there is a nbhd U of p in $\mathbb{R}^n$ and a smooth map

$g: U \rightarrow \mathbb{R}^n$ such that

$$g(x) = f(x) \text{ for all } x \in X \cap U.$$



Def: A map $f: X \rightarrow Y$ between manifolds is called a diffeomorphism if f is bijective and $f^{-1}: Y \rightarrow X$ is smooth.

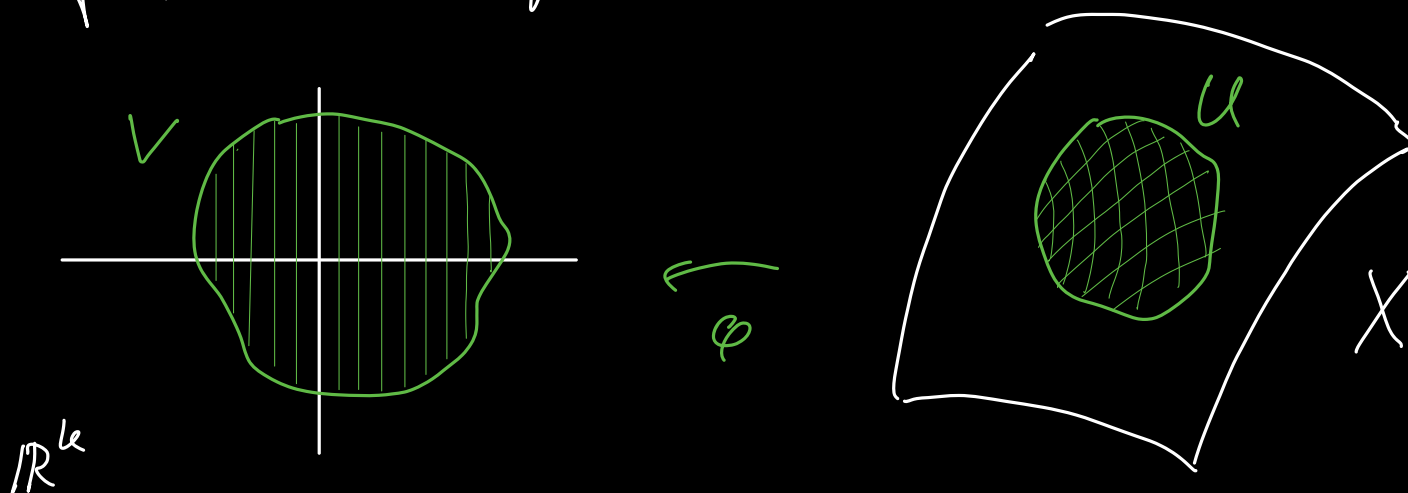
From now on: manifold = submfd of some $\mathbb{R}^n$.

Def: Let X be a k -dim. manifold and $p \in X$. By a chart for X around p we mean a diffeomorphism

$$\varphi: U \rightarrow V$$

from a nbhd of p in X onto an open set $V \subset \mathbb{R}^k$. The inverse map

$\varphi^{-1}: V \rightarrow U$ is called a local parametrization of X around p .



Notation for chart: (U, φ)

U = chart domain.

Obv: Any submfd chart is a chart.

Def: Let $S \subset \mathbb{R}^n$ be a k -dim. submfd and $p \in S$. A vector $v \in \mathbb{R}^n$ is called a tangent vector to S at p if there is a smooth curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow S$ such that

$$\gamma(0) = p, \quad (\iota \circ \gamma)'(0) = v,$$

where $\iota: S \rightarrow \mathbb{R}^n$ is the inclusion map.

$$\iota \circ \gamma: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$$

$T_p S := \{ \text{tangent vectors to } S \text{ at } p \}$

is called the tangent space to S at p .

Prop: $T_p S$ is a k -dim. linear subspace of $\mathbb{R}^n$.

Proof: Choose a chart (U, α) around p and let $V := \alpha(U) \subset \mathbb{R}^k$, $q := \alpha(p)$.

$\psi := \alpha^{-1}: V \rightarrow U$. We will show that

* $d\psi_q: \mathbb{R}^k \rightarrow \mathbb{R}^n$ is injective,

* $T_p S = \text{im}(d\psi_q)$.

(i) $d\psi_{\sharp}$ injective: By assumption there exist a subset W of U in $\mathbb{R}^n$ and a smooth map $\Phi: W \rightarrow \mathbb{R}^k$ such that

$$\Phi = \varphi \text{ on } S \cap W.$$

Then

$$\Phi \circ \psi = \text{id}_V \Rightarrow d\Phi_p \cdot d\psi_{\sharp} = \text{id}_{\parallel}$$

$$\Rightarrow d\psi_{\sharp} \text{ injective.}$$

(ii) Let $v \in T_p S$. Then there is a

smooth curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow S$ such that

$$\gamma(0) = p, \quad \gamma'(0) = v.$$

$$\lambda = \psi \circ \varphi \circ \gamma$$

$$v = \gamma'(0) = d\psi_{\sharp} \left(\underbrace{(\varphi \circ \gamma)'(0)}_{\in \mathbb{R}^n} \right) \in \text{im}(d\psi_{\sharp})$$

(iii) Let $w \in \mathbb{R}^n$. Choose $\varepsilon > 0$ such that

$$q + tw \in V \quad \text{for } |t| < \varepsilon$$

Define $\gamma: (-\varepsilon, \varepsilon) \rightarrow S$,

$$\gamma(t) = \psi(q + tw)$$

$$\gamma'(0) = d\psi_q(w)$$

$$\uparrow$$
$$T_p S$$